Symbolic dynamics: from the N-centre to the $(N+1)$-body problem, a preliminary study

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Abstract

We consider a rotating N-centre problem, with $N \geq 3$ and homogeneous potentials of degree $-\alpha < 0$, $\alpha \in [1,2)$. We prove the existence of infinitely many collision-free periodic solutions with negative and small *Jacobi constant* and small values of the angular velocity, for any initial configuration of the centres. We will introduce a Maupertuis' type variational principle in order to apply the *broken geodesics technique* developed in [11]. Major difficulties arise from the fact that, contrary to the classical Jacobi length, the related functional does not come from a Riemaniann structure but from a Finslerian one. Our existence result allows us to characterize the associated dynamical system with a symbolic dynamics, where the symbols are given partitions of the centres in two non-empty sets.

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1 Introduction and main results

In the classical N-centre problem it is investigated the motion of a test particle of null mass under the gravitational force fields of N fixed heavy bodies (the *centres*): if c_k and m_k denote respectively the position and the mass of the k-th centre, the motion equation is

$$\ddot{x}(t) = -\sum_{k=1}^{N} \frac{m_k}{|x(t)-c_k|^3}(x(t)-c_k) = \nabla \left(\sum_{k=1}^{N} \frac{m_k}{|x-c_k|} \right) \Bigg|_{x=x(t)}, \tag{1}$$

where $x = x(t) \in \mathbb{R}^2$ denotes the position of the particle at time $t \in \mathbb{R}$; basic references for such a problem are [3, 4, 6, 7, 8, 9, 11] and the references therein. In this paper we consider α-gravitational potentials of type

$$V(x) = \sum_{k=1}^{N} \frac{m_k}{\alpha |x-c_k|^\alpha} \qquad \alpha \in [1,2).$$

Of course, for $\alpha = 1$ we get the classical Newtonian potential; moreover, we assume that the centres are not fixed, but rotate according to the law $\xi_k(t) := \exp\{i\nu t\}c_k$. Under this assumption, the equation for the motion of the test particle is

$$\ddot{x}(t) = -\sum_{k=1}^{N} \frac{m_k}{|x(t) - e^{i\nu t}c_k|^{\alpha+2}} \left(x(t) - e^{i\nu t}c_k\right). \tag{2}$$

We will refer to the research of solutions to this equation as to *the rotating N-centre problem* (briefly, the rotating problem). It is convenient to introduce a different frame of reference for x, taking into account the rotation of the centres: setting $x(t) = \exp\{i\nu t\}z(t)$, equation (2) becomes

$$\ddot{z}(t) + 2\nu i\dot{z}(t) = \nu^2 z(t) - \sum_{k=1}^{N} \frac{m_k}{|z(t) - c_k|^{\alpha+2}} \left(z(t) - c_k\right). \tag{3}$$

We introduce $\Phi_\nu(z) := \nu^2|z|^2/2 + V(z)$, so that (3) can be written as

$$\ddot{z}(t) + 2\nu i\dot{z}(t) = \nabla\Phi_\nu(z(t)).$$

Since the terms in z and $\dot{z}$ are multiplied by powers of ν, the idea is that if $|\nu|$ is sufficiently small, then equation (3) can be regarded as a perturbation of the planar N-centre problem, which we dealt with in [11]. Note that, contrary to (1), equation (3) is not a conservative system; however, it is possible to find a first integral defining

$$J_\nu(z, \dot{z}) := \frac{1}{2}|\dot{z}|^2 - \Phi_\nu(z).$$

The value $h = J_\nu(z(t), \dot{z}(t))$, which is the same for every $t \in I$, is called the *Jacobi constant*, in analogy with the same integral of the circular restricted $(N+1)$-body problem (see the discussion below for the relationship between the rotating problem and the restricted one). Note the similarity between J_ν and the usual energy function $H(z, \dot{z}) = |\dot{z}|^2/2 - V(z)$: it results $H = J_0$.
In this paper we generalize the approach already developed in [11], proving the existence of infinitely many collision-free periodic solutions of equation (3) with negative and small (in absolute value) Jacobi constant, provided the angular velocity $|\nu|$ is sufficiently small. As a consequence, for those values of h and ν we can characterize the dynamical system induced by (3) on the level set

$$\mathcal{U}_{h,\nu} := \left\{(z, v) \in \mathbb{R}^4 : J_\nu(z, v) = h\right\}$$

with a symbolic dynamics, where the symbols are some selected partitions of the centres in two different non-empty sets. Coming back to equation (2), this means that for $h < 0$ and $|h|, |\nu|$ sufficiently small we have infinitely many collision-free *relative periodic solutions* (i.e. periodic in the rotating frame of reference) of the rotating problem; this existence result allows to prove the occurrence of symbolic dynamics in a proper submanifold of the phase space (which correspond to $\mathcal{U}_{h,\nu}$ through the transformation $x \leftrightsquigarrow z$).

Motivations. The N-centre problem can be considered as a simplified version of the $(N+1)$-body problem, when one of the bodies is much faster then the others. Therefore, in order to understand if the *broken geodesics technique* we introduced in [11] can be somehow extended to find solutions of the $(N+1)$-body problem, it seems reasonable to start considering an "easy test motion" for

the centres, such as the uniformly circular one. This is strictly related to the study of the circular restricted $(N+1)$-body problem, which we briefly recall; assigned N positive masses $m_1, \ldots, m_N$, let us consider any planar central configuration $(c_1, \ldots, c_N)$ of the N-body problem. A relative equilibrium of the N-body problem is a motion of type $\xi_k(t) := \exp\{i\nu t\} c_k$ $(k = 1, \ldots, N)$, with $\nu \in \mathbb{R}$, i.e. an equilibrium point in a rotating frame of reference with angular velocity ν. The restricted problem consists in studying the motion of a test particle of null mass under the gravitational force field of N bodies (the *primaries*) which move according to a motion of relative equilibrium. This leads to the search of solutions to (3), but now ν cannot be considered as a free parameter: indeed, each central configuration determines the unique admissible value of ν through the relation

$$\nu^2 = \frac{U(\mathbf{c})}{2I(\mathbf{c})}, \quad \text{where} \quad U(\mathbf{c}) = \sum_{1 \le j < k \le N} \frac{m_j m_k}{|c_j - c_k|}, \; I(\mathbf{c}) = \frac{1}{2} \sum_{k=1}^{N} m_k |c_k|^2, \tag{4}$$

see Meyer [10]. In particular, letting ν to tend to 0, the relation (4) implies that either $m_k \to 0$ for every k or $|c_k| \to 0$ for every k; as a consequence, the equation of the restricted problem in the limit case $\nu \to 0$ tends to $\ddot{z} = 0$, which has no relation with the N-centre problem or the N-body problem. As a toy model towards the real restricted $(N+1)$-body problem, we introduce the rotating N-centre problem; we point out that the motivation for its study is prevalently mathematical: our goal is to understand if the techniques introduced in [11] are sufficiently robust to survive when we perturb the N-centre problem by letting the centres move; the answer is yes, but, as we will see, the extension of our broken geodesics method is not trivial and requires new ideas. Therefore, the generalization to the real restricted problem seems possible, but extremely complicated.

1.1 Periodic solutions

Let $\mathcal{P}$ be the set of the possible partitions of the centers in two different non-empty sets. There are exactly $2^{N-1} - 1$ such partitions, and to each of them we associate a label:

$$\mathcal{P} = \left\{ P_j : j = 1, \ldots, 2^{N-1} - 1 \right\}.$$

We give particular labels to those partitions which isolates one centre with respect to the others:

$$P_j := \{\{c_j\}, \{c_1, \ldots, c_N\} \setminus \{c_j\}\} \qquad j = 1, \ldots, N.$$

The collection of these labels is the subset

$$\mathcal{P}_1 := \{P_j \in \mathcal{P} : j = 1, \ldots, N\} \subset \mathcal{P}. \tag{5}$$

We define the *right shift* $T_r : \mathcal{P}^n \to \mathcal{P}^n$ as

$$T_r((P_{j_1}, P_{j_2}, \ldots, P_{j_n})) = (P_{j_n}, P_{j_1}, \ldots, P_{j_{n-1}}),$$

and we say that $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$ *is equivalent to* $(P'_{j_1}, \ldots, P'_{j_n}) \in \mathcal{P}^n$ if there exists $m \in \mathbb{N}$ such that

$$(P'_{j_1}, \ldots, P'_{j_n}) = T_r^m \left((P_{j_1}, \ldots, P_{j_n}) \right).$$

To describe the first main result which we are going to prove, let us look at Theorem 1.1 of [11]; therein we proved the existence of $\bar{h} < 0$ such that for any $h \in (\bar{h}, 0)$ we can associate to any finite sequence of partition $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$ a periodic solution $x_{((P_{j_1}, \ldots, P_{j_n}), h)}$ of the N-centre problem

(1) with energy h. Under particular assumptions on $(P_{j_1},\dots,P_{j_n})$, assumptions which are specified in points (*ii*)-*b*) or (*ii*)-*c*) of the quoted statement, we have to allow collision solutions, but it is always possible (for every $N\geq 3$) to build infinitely many collision-free solutions. We would like to repeat the game associating to a finite sequence of partitions, for sufficiently small values of the absolute value of the Jacobi constant $|h|$ and of the angular velocity $|\nu|$, a periodic solution of equation (3). In this paper we will put some restrictions on the sequences of partitions which we want to consider; this is motivated by the fact that the rotation of the centres makes impossible the use of some techniques employed in the study of the behaviour of collision-solutions. In this sense we observed in [11] that the study of the collisions requires a distinction among

$$1)\ \alpha=1 \text{ and } N\geq 4,\qquad 2)\ \alpha=1 \text{ and } N=3,\qquad 3)\ \alpha\in(1,2).$$

We start from the first case.

Theorem 1.1. *Let $\alpha=1$, $N\geq 4$, $c_1,\dots,c_N\in\mathbb{R}^2$, $m_1,\dots,m_N\in\mathbb{R}^+$. There exists $\bar h_1$ such that, given $h\in(\bar h_1,0)$, there is $\bar\nu_1=\bar\nu_1(h)>0$ such that to each $\nu\in(-\bar\nu_1,\bar\nu_1)$, $n\in\mathbb{N}$ and $(P_{j_1},\dots,P_{j_n})\in(\mathcal{P}\setminus\mathcal{P}_1)^n$ we can associate a collision-free periodic solution $z_{((P_{j_1},\dots,P_{j_n}),h,\nu)}$ of*

$$\begin{cases}\ddot z(t)+2\nu i\dot z(t)=\nabla\Phi_\nu(z(t))\\ \frac{1}{2}|\dot z(t)|^2-\Phi_\nu(z(t))=h,\end{cases}\tag{6}$$

which depends on $(P_{j_1},\dots,P_{j_n})$ in the following way. There exist $\bar R,\bar\delta>0$ (depending on h only) such that $z_{((P_{j_1},\dots,P_{j_n}),h,\nu)}$ crosses $2n$ times within one period the circle $\partial B_{\bar R}(0)$, at times $(t_k)_{k=0,\dots,2n-1}$, and

- *in (t_{2k},t_{2k+1}) the solution stays outside $B_{\bar R}(0)$ and*

$$|z_{((P_{j_1},\dots,P_{j_n}),h,\nu)}(t_{2k})-z_{((P_{j_1},\dots,P_{j_n}),h,\nu)}(t_{2k+1})|<\bar\delta.$$

- *in (t_{2k+1},t_{2k+2}) the solution lies inside $B_{\bar R}(0)$ and separates the centres according to the partition P_{j_k}.*

Note the analogy with Theorem 1.1 of [11]: if $\alpha=1$ and $N\geq 4$ we can easily find a condition on $(P_{j_1},\dots,P_{j_n})$ in order to ensure that the periodic solution $z_{((P_{j_1},\dots,P_{j_n}),h,0)}$ of the N-centre problem

$$\begin{cases}\ddot z(t)=\nabla V(z(t))\\ \frac{1}{2}|\dot z(t)|^2-V(z(t))=h\end{cases}$$

is collision-free; it is sufficient to impose that $P_{j_k}\in(\mathcal{P}\setminus\mathcal{P}_1)$ for every k. If $N=3$ then $\mathcal{P}=\mathcal{P}_1$, so that if in addition $\alpha=1$ we have to use a little trick: let

$$(P_1,P_1,P_2,P_3)=G_1,\qquad (P_2,P_2,P_3,P_1)=G_2,$$

and let $\mathcal{G}:=\{G_1,G_2\}$. We will observe (Remark 13 below) that no composed sequence obtained by the juxtaposition of G_1 and G_2 satisfies the symmetry conditions of cases (*ii*)-*b*) or (*ii*)-*c*) of Theorem 1.1 in [11]; this implies that a solution of the N-centre problem associated to $(P_{k_1},\dots,P_{k_{4n}})\in\mathcal{G}^n\subset\mathcal{P}^{4n}$ is collision-free. Coming back to the rotating problem, it results

Theorem 1.2. *Replacing the assumption $N\geq 4$ in Theorem 1.1 with $N=3$, the same statement holds true replacing $(\mathcal{P}\setminus\mathcal{P}_1)^n$ with $\mathcal{G}^n$.*

If $\alpha \neq 1$ this is not necessary, since in such a case $z_{((P_{j_1},\dots,P_{j_n}),h,0)}$ was proved to be always collision-free.

Theorem 1.3. *Replacing the assumptions $\alpha = 1$ and $N \geq 4$ with $\alpha \in (1, 2)$ and $N \geq 3$, the previous statement holds true, replacing the set $\mathcal{P} \setminus \mathcal{P}_1$ with $\mathcal{P}$.*

Remark 1. The assumption "$|h|$ is sufficiently small" is substantial, as we can immediately realize observing that if z is a solution of (6), then the curve parametrized by z in the configuration space has to be confined in $\{\Phi_\nu(z) \geq -h\}$. If $h < 0$ becomes large in absolute value, we obtain a disconnected set, so that to find solutions exhibiting the behavior described in the previous statements becomes impossible.

1.2 Symbolic dynamics

Similarly to Corollary 1.3 of [11], as a consequence of Theorem 1.1, 1.2, 1.3, we obtain the following result.

Corollary 1.4. *Let $\alpha \in [1, 2)$, $N \geq 3$, $m_1, \dots, m_N \in \mathbb{R}^+$ and $c_1, \dots, c_N \in \mathbb{R}^2$. Let $h \in (\bar{h}_1, 0)$ and $\nu \in (-\bar{\nu}_1(h), \bar{\nu}_1(h))$, where $\bar{h}_1$ and $\bar{\nu}_1(h)$ have been introduced in Theorem 1.1, 1.2, 1.3. There exists a subset $\Pi_{h,\nu}$ of the level set $\mathcal{U}_{h,\nu}$, a return map $\mathfrak{R} : \Pi_{h,\nu} \to \Pi_{h,\nu}$ for the dynamical system associated to equation (3), a set of symbols $\widehat{\mathcal{P}}$ and a continuous and surjective map $\pi : \Pi_{h,\nu} \to \widehat{\mathcal{P}}^{\mathbb{Z}}$, such that the diagram*

$$\begin{tikzcd} \Pi_{h,\nu} \arrow[r, "\mathfrak{R}"] \arrow[d, "\pi"] & \Pi_{h,\nu} \arrow[d, "\pi"] \\ \widehat{\mathcal{P}}^{\mathbb{Z}} \arrow[r, "T_r"] & \widehat{\mathcal{P}}^{\mathbb{Z}}, \end{tikzcd}$$

commutes (here T_r demotes the right shift in $\widehat{\mathcal{P}}^{\mathbb{Z}}$); namely for every $h \in (\bar{h}_1, 0)$ and $\nu \in (-\bar{\nu}_1(h), \bar{\nu}_1(h))$, the restriction of the dynamical system associated to the rotating problem on the level set $\mathcal{U}_{h,\nu}$ has a symbolic dynamics.

1.3 Plan of the paper

We follow here the same general strategy already developed for proving Theorem 1.1 of [11]. In Section 2 we will perform a suitable rescaling in order to pass from problem (6) to an equivalent problem where the parameter Jacobi constant will be replaced by the parameter given by the maximal distance of the centres from the origin. This leads to the study of a rotating problem with a rescaled potential

$$V_\varepsilon(y) = \sum_{k=1}^{N} \frac{m_k}{|y - c'_k|^\alpha} \quad \text{where} \quad \max_{1 \leq k \leq N} |c'_k| = \varepsilon, \tag{7}$$

and a different angular velocity ν'; we will be interested in solutions with Jacobi constant equal to -1. In this way, outside a ball or radius $R > \varepsilon > 0$, and for $|\nu'|$ sufficiently small, the equivalent problem

$$\begin{cases} \ddot{y}(t) + 2\nu' i \dot{y}(t) = \nabla \left(\frac{(\nu')^2}{2} |y|^2 + V_\varepsilon(y) \right) \\ \frac{1}{2}|\dot{y}(t)|^2 - \frac{(\nu')^2}{2}|y(t)|^2 - V_\varepsilon(y(t)) = -1 \end{cases} \tag{8}$$

is a small perturbation of the Kepler's problem with homogeneity degree $-\alpha < 0$, $\alpha \in [1,2)$. This is why we will face the research of periodic solutions of (8) splitting the study of the dynamics outside/inside a ball $B_R(0)$ (R will be conveniently chosen). As in [11], outside $B_R(0)$ we will find arcs of solutions of (8) connecting two points $p_0, p_1 \in \partial B_R(0)$, provided their distance is sufficiently small, via perturbative techniques. With respect to [11], we have to take into account the new parameter ν', but the argument is substantially the same.
In section 4 we study the problem inside $B_R(0)$, trying again to follow the line of reasoning of [11]; we will search minimizers of the Jacobi type functional

$$L_{h,\nu} := \int_0^1 |\dot{u}|\sqrt{\Phi_\nu(u) - 1} + \frac{\nu}{\sqrt{2}}\int_0^1 \langle iu, \dot{u}\rangle$$

under suitable constraints, in order to connect any pair $p_1, p_2 \in \partial B_R(0)$ with arcs of solution of (8) which separate the centres according to any prescribed partition in $\mathcal{P}$. The functional $L_{h,\nu}$, contrary to the classical Jacobi length, does not come from a Riemaniann structure but from a Finslerian one. A main consequence is the lack of reversibility of the problem, and this marks a significant difference in the argument to rule out the possibility of having collisions for its minimizers. The alternative "collision less" or "ejection-collision", valid for the N-centre problem, does not holds anymore. This is why we will need an "ad hoc" argument, which will be exposed in sections 6 and 7.
The collection of the outer and inner dynamics will be the object of section 5. Assigned a sequence $(P_{j_1}, \dots, P_{j_n}) \in \mathcal{P}^n$ and ε and ν' sufficiently small, the aim will be the construction of a weak periodic solution $y_{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ of the restricted problem crossing $2n$ times within one period the circle $\partial B_R(0)$, at times $(t_k)_{k=0,\dots,2n-1}$, and

- in (t_{2k}, t_{2k+1}) the solution stays outside $B_R(0)$ and
$$|y_{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu)}(t_{2k}) - y_{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu)}(t_{2k+1})| < \bar{\delta}.$$
- in (t_{2k+1}, t_{2k+2}) the solution lies inside $B_{\bar{R}}(0)$ and parametrizes an inner local minimizer of the functional $L_{-1,\nu'}$ which, up to collisions, separates the centres according to the partition P_{j_k}.

This will be achieved glueing the fixed ends trajectories found in sections 3 and 4, alternating outer and inner arcs. In order to obtain smooth junctions, we are going to use the variational argument already carried on in [11] with success.
Finally, in sections 6 and 7, we will complete the proof of Theorems 1.1, 1.2 and 1.3, providing sufficient conditions on the sequences $(P_{j_1}, \dots, P_{j_n})$ in order to have collision-free solutions; this will be done through a kind of Gamma-convergence argument: we will see that the minimizers of $L_{-1,\nu'}$ are weakly convergent in H^1, as $\nu' \to 0$, to the minimizers of $L_{-1,0}$, which is the classical Jacobi functional. Therefore we will exploit the description of the behaviour of such minimizers given in [11].

2 Preliminaries

Let us fix $N \geq 3$, $\alpha \in [1,2)$, $c_1, \dots, c_N \in \mathbb{R}^2$ and $m_1, \dots, m_N > 0$, and let $M = \sum_{k=1}^N m_k$; we fix the origin in the centre of mass. In this section we prove that to find a periodic solution of the rotating problem (3) with Jacobi constant $h < 0$ is equivalent to find a periodic solution of a different rotating problem with Jacobi constant equal to -1. In this perspective the maximal distance of the centres from the origin replaces h as parameter, and the angular velocity changes as well. To be precise one can easily prove:

Proposition 2.1. *Let $z \in \mathcal{C}^2\left((a,b);\mathbb{R}^2\right)$ be a classical solution of (3) with Jacobi constant $h<0$. Then the function*

$$y(t) = (-h)^{\frac{1}{\alpha}}\, z\left((-h)^{-\frac{\alpha+2}{2\alpha}}\, t\right), \qquad t \in \left((-h)^{\frac{\alpha+2}{2\alpha}}\, a, (-h)^{\frac{\alpha+2}{2\alpha}}\, b\right) \tag{9}$$

is a solution of a rotating problem with

$$c_j' = (-h)^{\frac{1}{\alpha}}\, c_j, \quad j=1,\dots,N \quad \text{and} \quad \nu' = (-h)^{-\frac{\alpha+2}{2\alpha}}\,\nu; \tag{10}$$

the Jacobi constant of y as solution of the new problem is -1. Conversely: let $y \in \mathcal{C}^2\left((a',b'),\mathbb{R}^2\right)$ be a classical solution with Jacobi constant -1 of a rotating problem with initial configuration of the centres $\{c_j'\}$ and angular velocity ν'. Let us set

$$c_j = (-h)^{-\frac{1}{\alpha}}\, c_j', \quad j=1,\dots,N \quad \text{and} \quad \nu = (-h)^{\frac{\alpha+2}{2\alpha}}\,\nu'.$$

Then

$$z(t) = (-h)^{-\frac{1}{\alpha}}\, y\left((-h)^{\frac{\alpha+2}{2\alpha}}\, t\right), \qquad t \in \left((-h)^{-\frac{\alpha+2}{2\alpha}}\, a', (-h)^{-\frac{\alpha+2}{2\alpha}}\, b'\right)$$

is a classical solution of (3) with Jacobi constant $h<0$.

Corollary 2.2. *For every $\varepsilon>0$ and for every $\widetilde{\nu}\in\mathbb{R}$ there exist $\zeta_1(\varepsilon)$ and $\zeta_2(\varepsilon,\widetilde{\nu})\in\mathbb{R}$ such that if $h=\zeta_1(\varepsilon)$ and $\nu=\zeta_2(\varepsilon,\widetilde{\nu})$ then*

$$\max_{1\le k\le N}|c_k'| = \varepsilon, \qquad \nu' = \widetilde{\nu}.$$

The function ζ_1 is strictly decreasing in ε, the function ζ_2 is strictly increasing both in ε and $\widetilde{\nu}$.

Remark 2. Problem (8) for $(\varepsilon,\nu')\in(0,\bar{\varepsilon})\times(-\bar{\nu}',\bar{\nu}')$ is equivalent, through Proposition 2.1 and Corollary 2.2, to equation (3) associated with Jacobi constant $h<0$ and angular velocity ν for $(h,\nu)\in(-\zeta_1(\bar{\varepsilon}),0)\times(-\zeta_2(\bar{\varepsilon},\bar{\nu}),\zeta_2(\bar{\varepsilon},\bar{\nu}))$. Two corresponding solutions exhibit the same topological behaviour, as showed by equation (9). Note that more the Jacobi constant is small, more the admissible angular velocities have to be small.

Let us fix $\varepsilon>0$, $\nu'\in\mathbb{R}$, and $K := \overline{B_{R_2}(0)}\setminus B_{R_1}(0)$, with $R_2>R_1>\varepsilon$. In K we can consider the new problem as a small perturbation of the α-Kepler's problem, whose potential is

$$V_0(y) := \frac{M}{\alpha|y|^\alpha} \qquad y\in\mathbb{R}^2\setminus\{0\}.$$

Indeed, setting

$$\Phi_{\nu',\varepsilon}(y) := \frac{(\nu')^2}{2}|y|^2 + V_\varepsilon(y),$$

(V_ε has been already defined in (7)), it is not difficult to check that

$$\|\Phi_{\nu',\varepsilon} - V_0\|_{\mathcal{C}^1(K)} = o(\varepsilon) + o(\nu') \qquad \text{for } \varepsilon\to 0^+,\ \nu'\to 0. \tag{11}$$

Let us observe that if y is a solution of $\ddot{y}+2\nu' i\dot{y} = \nabla\Phi_{\nu',\varepsilon}(y)$ with Jacobi constant -1 over an interval $I\subset\mathbb{R}$, then

$$\Phi_{\nu',\varepsilon}(y(t)) \ge 1 \qquad \forall t\in I.$$

To exploit the perturbative nature of the problem outside a ball $B_R(0)$, we have to check that, for $\varepsilon>0$ sufficiently small and for ν' in a neighbourhood of 0, there exists $R>0$ such that

$$B_\varepsilon(0)\subset B_R(0)\subset\left\{y\in\mathbb{R}^2 : \Phi_{\nu',\varepsilon}(y)\ge 1\right\}. \tag{12}$$

Then, considering any compact set $B_R(0)\subset K\subset\{\Phi_{\nu',\varepsilon}(y)\ge 1\}$, we will be able to use (11) in $K\setminus B_R(0)$.

Proposition 2.3. *Let $\varepsilon > 0$, $\nu' \in \mathbb{R}$. Let $R > 0$ such that $\varepsilon < R < \left(\frac{M}{\alpha}\right)^{1/\alpha} - \varepsilon$. Then (12) holds true. There exists $\varepsilon_1 > 0$ such that, for every $0 < \varepsilon < \varepsilon_1$, this choice is possible.*

Actually, we will make the further request $\varepsilon < R/2 < R < \left(\frac{M}{\alpha}\right)^{1/\alpha} - \varepsilon$. which is satisfied for every $\varepsilon \in (0, \varepsilon_1/2)$. As in [11], we select R so that $\partial B_R(0)$ is the image of the circular solution of the α-Kepler's problem with energy -1:

$$R := \left(\frac{(2-\alpha)M}{2\alpha}\right)^{\frac{1}{\alpha}}. \tag{13}$$

This is consistent with the previous restriction on R, if ε_1 is sufficiently small (if this was not true, it is sufficient to replace ε_1 with a smaller quantity).

Remark 3. For future convenience, note that for every $y \in \overline{B_R(0)}$

$$V_\varepsilon(y) - 1 \geq \frac{M}{\alpha\left(\left(\frac{(2-\alpha)M}{2\alpha}\right)^{\frac{1}{\alpha}} + \varepsilon\right)^\alpha} - 1 \geq \frac{M}{\alpha\left(\left(\frac{(2-\alpha)M}{2\alpha}\right)^{\frac{1}{\alpha}} + \varepsilon_1\right)^\alpha} - 1 =: M_1 > 0, \tag{14}$$

and hence $\Phi_{\nu',\varepsilon}(y) - 1 \geq M_1$. This value is independent on $\varepsilon \in (0, \varepsilon_1/2)$. From now on we will use M_1 to denote this positive constant.

3 Outer dynamics

We are going to use a perturbative approach in order to find solutions of

$$\begin{cases} \ddot{y}(t) + 2\nu' i\dot{y}(t) = \nabla\Phi_{\nu',\varepsilon}(y(t)) & t \in [0,T] \\ \frac{1}{2}|\dot{y}(t)|^2 - \Phi_{\nu',\varepsilon}(y(t)) = -1 & t \in [0,T] \\ |y(t)| > R & t \in (0,T) \\ y(0) = p_0 \qquad y(T) = p_1 & \end{cases} \tag{15}$$

when the distance between $p_0, p_1 \in \partial B_R(0)$ is sufficiently small; T has to be determined. To be precise we will prove the following Proposition.

Proposition 3.1. *There exist $\delta > 0$, $\varepsilon_2 > 0$ and $\nu'_1 > 0$ such that for every $(\varepsilon, \nu') \in (0, \varepsilon_2) \times (-\nu'_1, \nu'_1)$, for every $p_0, p_1 \in \partial B_R(0) : |p_1 - p_0| < 2\delta$, there exist a unique solution $y_{ext}(\cdot\,; p_0, p_1; \varepsilon, \nu')$ of (15) with $T = T_{ext}(p_0, p_1; \varepsilon, \nu') > 0$. This solution depends in a $\mathcal{C}^1$ way on the endpoints p_0 and p_1, and*

$$\begin{aligned} \max_{t\in[0,T_{ext}(p_0,p_1;\varepsilon,\nu')]} |y_{ext}(t; p_0, p_1; \varepsilon, \nu')| &\leq 2\left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} \\ \max_{t\in[0,T_{ext}]} |\dot{y}_{ext}(t; p_0, p_1; \varepsilon, \nu')| &\leq 2\sqrt{2\left(-1 + \frac{M}{\alpha R^\alpha}\right)} \end{aligned} \tag{16}$$

for every $(p_0, p_1) \in \{(p_0, p_1) \in (\partial B_R(0))^2 : |p_0 - p_1| < 2\delta\}$, $\varepsilon \in (0, \varepsilon_2)$ and $\nu' \in (-\nu'_1, \nu'_1)$.

We will follow the same line of reasoning of the proof of Theorem 3.1 of [11], with the only difference that here we add the parameter ν'. For the reader's convenience, we will review the main steps. For

every $p_0 = R\exp\{i\vartheta_0\} \in \partial B_R(0)$, the unperturbed problem ($\varepsilon = 0$ and $\nu' = 0$) is

$$\begin{cases} \ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} & t \in [0,T] \\ \frac{1}{2}|\dot{y}(t)|^2 - \frac{M}{\alpha|y(t)|^{\alpha}} = -1 & t \in [0,T] \\ |y(t)| > R & t \in (0,T) \\ y(0) = p_0, \qquad y(T) = p_0. \end{cases} \tag{17}$$

Let us solve the Cauchy problem

$$\begin{cases} \ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} \\ y(0) = p_0, \qquad\qquad \dot{y}(0) = \sqrt{2\left(-1+\frac{M}{\alpha R^{\alpha}}\right)}\left(\frac{p_0}{R}\right). \end{cases}$$

The trajectory returns at the point p_0 after a certain time $\bar{T} > 0$, having swept the portion of the rectilinear brake orbit of energy -1 starting from p_0 and lying in $\mathbb{R}^2 \setminus B_R(0)$. Our aim is to catch the behaviour of the solutions under small variations of the initial conditions. We consider

$$\begin{cases} \ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} \\ y(0) = p_0, \qquad \dot{y}(0) = \dot{r}_0 e^{i\vartheta_0} + R\dot{\vartheta}_0 i e^{i\vartheta_0}, \end{cases} \tag{18}$$

where $\dot{r}_0$ is assigned as function of $\dot{\vartheta}_0$ by means of the energy integral. We denote as $y(\cdot\,;\vartheta_0,\dot{\vartheta}_0)$ the solution of (18). For the brake orbit $y(\cdot\,;\vartheta_0,0)$ it results

$$\vartheta(t;\vartheta_0,0) \equiv \vartheta_0 \qquad \forall t \in [0,\bar{T}].$$

We introduce $\psi : \Theta \times I \to \mathbb{R}^2$ as

$$\psi(\dot{\vartheta}_0, T) := y(T;\vartheta_0,\dot{\vartheta}_0),$$

where $\Theta \times I \subset S^1 \times \mathbb{R}$ is a neighbourhood of $(0,\bar{T})$ on which ψ is well defined. The following result is Lemma 3.2 of [11], see the proof therein.

Lemma 3.2. *The Jacobian of ψ in $(0,\bar{T})$ is invertible.*

Now we introduce the parameters ε and ν': let us define

$$\begin{aligned} \Psi :& \Theta \times I \times \partial B_R(0) \times \left[0, \frac{\varepsilon_1}{2}\right) \times \mathbb{R} \to \mathbb{R}^2 \\ & (\dot{\vartheta}_0, T, p_1, \varepsilon, \nu') \mapsto y(T;\vartheta_0,\dot{\vartheta}_0;\varepsilon,\nu') - p_1, \end{aligned}$$

where $y(\cdot\,;\vartheta_0,\dot{\vartheta}_0;\varepsilon,\nu')$ is the solution of

$$\begin{cases} \ddot{y}(t) + 2\nu' i\dot{y}(t) = \nabla\Phi_{\nu',\varepsilon}(y(t)) \\ y(0) = p_0, \qquad \dot{y}(0) = \dot{r}_{\nu',\varepsilon} e^{i\vartheta_0} + R\dot{\vartheta}_0 i e^{i\vartheta_0}, \end{cases} \tag{19}$$

and $\dot{r}_{\nu',\varepsilon}$ is assigned as function of $\dot{\vartheta}_0, \varepsilon, \nu'$ by means of the Jacobi constant. The proof of the following statement is a straightforward generalization of the proof of Lemma 3.3 in [11].

Lemma 3.3. *There exist $\delta > 0$, $0 < \varepsilon_2 < \varepsilon_1/2$ and $\nu_1' > 0$ such that for every $(\varepsilon,\nu') \in (0,\varepsilon_2) \times (-\nu_1',\nu_1')$, for every $p_1 \in \partial B_R(0) : |p_1 - p_0| < 2\delta$, there exists a unique solution $y(\cdot\,;\vartheta_0,\dot{\vartheta}_0;\varepsilon,\nu')$ of (19) defined in $[0,T]$ for a certain $T > 0$, and satisfying also (15). Moreover, it is possible to choose δ, ε_2 and ν_1' independent on $p_0 \in \partial B_R(0)$.*

Proposition 3.1 follows. The solutions obtained are uniquely determined and depends in a smooth way on the ends p_0 and p_1, and on the parameters ε and ν' (by the implicit function Theorem). Since a brake solution $y_{\rm br}(\cdot) = y(\cdot\,; p_0, p_0; 0, 0)$ of the Kepler's problem is such that

$$\max_{t\in[0,\bar T]} |y_{\rm br}(t)| = \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} \quad \text{and} \quad \max_{t\in[0,\bar T]} |\dot y_{\rm br}(t)| = \sqrt{2\left(-1 + \frac{M}{\alpha R^\alpha}\right)},$$

it is possible, if necessary, to replace ε_2 and ν'_1 with smaller quantities in such a way that (16) is satisfied.

Definition 1. For any $\varepsilon \in (0, \varepsilon_2)$ we pose

$$\mathcal{OS}_\varepsilon := \{y_{\rm ext}(\cdot\,; p_0, p_1; \varepsilon, \nu') : \; p_0, p_1 \in \partial B_R(0), \; |\nu'| < \nu'_1\},$$

i.e. $\mathcal{OS}_\varepsilon$ is the set of the *outer solutions* corresponding to a fixed value of ε.

Lemma 3.4. *For every $\varepsilon \in (0, \varepsilon_2)$ there exist $C_1, C_2 > 0$ such that*

$$C_1 \le T_{ext}(p_0, p_1; \varepsilon, \nu') \le C_2 \qquad \forall (p_0, p_1, \nu') \in (\partial B_R(0))^2 \times (-\tilde\nu', \tilde\nu').$$

Also, there exists $C_3 > 0$ such that

$$\|y_{ext}(\cdot\,; p_0, p_1; \varepsilon, \nu')\|_{H^1([0, T_{ext}(p_0,p_1;\varepsilon,\nu')])} \le C_3$$

for every $(p_0, p_1, \nu') \in (\partial B_R(0))^2 \times (-\tilde\nu', \tilde\nu')$.

Proof. The boundedness of $T_{\rm ext}(p_0, p_1; \varepsilon, \nu')$ is a consequence of the continuous dependence of the solutions with respect to variations of initial data. As far as the bound in the H^1 norm is concerned, we can use (16) and the first part. $\square$

Remark 4. We could make the boundedness properties described above uniform in ε. But we will use this Lemma in sections 5, 6 and 7, where ε will be fixed.

4 Inner dynamics

In contrast with the previous one, this section is not a direct generalization of section 4 of [11]; however, it is convenient to summarize the main ideas that we developed therein. Our goal was to find solutions of

$$\begin{cases} \ddot y(t) = \nabla V_\varepsilon(y(t)) & t \in [0, T] \\ \frac{1}{2}|\dot y|^2 - V_\varepsilon(y(t)) = -1 & t \in [0, T] \\ |y(t)| < R & t \in (0, T) \\ y(0) = p_1, \qquad y(T) = p_2. \end{cases} \tag{20}$$

satisfying particular topological requirements; T was not determined a priori, while the energy was fixed to -1; hence, in order to give a variational formulation of (20), it was convenient to adopt the Maupertuis' principle rather then the minimal action principle. Let $[a, b] \subset \mathbb{R}$ and $p_1, p_2 \in \partial B_R(0)$, $p_1 = R\exp\{i\vartheta_1\}$, $p_2 = R\exp\{i\vartheta_2\}$ (the case $p_1 = p_2$ is admissible). We introduced the set of collision-free H^1 paths

$$\begin{aligned} \widehat H_{p_1p_2}([a,b]) := \big\{ u \in H^1\left([a,b], \mathbb{R}^2\right) : u(a) = p_1, \; u(b) = p_2, \\ u(t) \neq c_j \text{ for every } t \in [a,b], \text{ for every } j \in \{1, \dots, N\} \big\}, \end{aligned} \tag{21}$$

the set of colliding H^1 functions

$$\mathfrak{Coll}_{p_1p_2}([a,b]) := \left\{ u \in H^1\left([a,b],\mathbb{R}^2\right) : u(a)=p_1,\ u(b)=p_2, \right. \\ \left. \exists t \in [a,b] : u(t)=c_j \text{ for some } j \in \{1,\dots,N\} \right\},$$

and their union

$$H_{p_1p_2}([a,b]) = \widehat{H}_{p_1p_2}([a,b]) \cup \mathfrak{Coll}_{p_1p_2}([a,b]).$$

Briefly, we will write $\widehat{H}$, $\mathfrak{Coll}$ and H when there will not be possibility of misunderstanding. Note that H is the closure of $\widehat{H}$ in the weak topology of H^1. A path $u \in \widehat{H}$ can be characterized according to its winding number with respect to each centre. This number can be computed by artificially closing the path itself, in the following way:

$$\Gamma(t) := \begin{cases} \begin{cases} u(t) & t \in [a,b] \\ Re^{i(t-b+\vartheta_2)} & t \in (b, b+\vartheta_1+2\pi-\vartheta_2) \end{cases} & \text{if } \vartheta_1 < \vartheta_2 \\ u(t) \qquad t \in [a,b] & \text{if } \vartheta_1 = \vartheta_2 \\ \begin{cases} u(t) & t \in [a,b] \\ Re^{i(t-b+\vartheta_2)} & t \in (b, b+\vartheta_1-\vartheta_2) \end{cases} & \text{if } \vartheta_1 > \vartheta_2, \end{cases}$$

i.e. if $p_1 \neq p_2$ we close the path u with the arc of $\partial B_R(0)$ connecting p_2 and p_1 in counterclockwise sense. Then it is well defined the usual winding number $\mathrm{Ind}\,(u([a,b]),c_j)$. Given $l=(l_1,\dots,l_N) \in \mathbb{Z}^N$, a connected component of $\widehat{H}$ is of the form

$$\widehat{\mathfrak{H}}_l^{p_1p_2}([a,b]) := \left\{ u \in \widehat{H}_{p_1p_2}([a,b]) : \mathrm{Ind}\,(u([a,b]),c_j) = l_j \quad \forall j=1,\dots,N \right\}.$$

We needed classes containing self-intersections-free paths, so that we considered $l \in \mathbb{Z}_2^N$ instead of $l \in \mathbb{Z}^N$, and set

$$\widehat{H}_l = \widehat{H}_l^{p_1p_2}([a,b]) := \left\{ u \in \widehat{H}_{p_1p_2}([a,b]) : \mathrm{Ind}\,(u([a,b]),c_j) \equiv l_j \mod 2 \quad \forall j=1,\dots,N \right\};$$

namely we collected together the components with winding numbers having the same parity with respect to each centre. We also assumed that

$$\exists j,k \in \{1,\dots,N\},\ j \neq k, \text{ such that } l_j \neq l_k \mod 2. \tag{22}$$

In this way, each $u \in \widehat{H}_l$ has to pass through the ball $B_\varepsilon(0)$, and cannot be constant even if $p_1=p_2$. Actually we proved that the functions in $\widehat{H}_l$ are uniformly non-constant, in the sense that there exists $C>0$ such that

$$\|\dot{u}\|_{L^2} \geq C \qquad \forall u \in \widehat{H}_l.$$

Furthermore, the constant C can be chosen independently on p_1 and p_2 (see Lemma 5.2 of the quoted paper) and also on l (the proof is the same). We said that $l \in \mathbb{Z}_2^N$ is a *winding vector*, and we term $\mathfrak{I}^N := \{l \in \mathbb{Z}_2^N : \text{l satisfies (22)}\}$. In order to apply variational methods, we needed to consider $H_l = H_l^{p_1p_2}([a,b])$, the closure of $\widehat{H}_l$ with respect to the weak topology of H^1; of course, in H_l there are collision-function. Since we searched functions whose images are in $B_R(0)$, we considered the subsets

$$\widehat{K}_l = \widehat{K}_l^{p_1p_2}([a,b]) := \{u \in \widehat{H}_l : |u(t)| \leq R \ \forall t \in [a,b]\}$$
$$K_l = K_l^{p_1p_2}([a,b]) := \{u \in H_l : |u(t)| \leq R \ \forall t \in [a,b]\}.$$

The set K_l is weakly closed in H^1. Recall the definition of the Maupertuis' functional associated to problem (20):

$$M_{-1}(u) = M_{-1}([a,b];u) := \frac{1}{2}\int_a^b |\dot{u}|^2 \int_a^b (V_\varepsilon(u) - 1)\,; \tag{23}$$

It is well known that solutions of the fixed energy problem given by the first two equations in (20) are obtained as re-parametrizations of critical points of M_{-1} at positive level in the space $\widehat{H}$ (see, e.g. [1]). It is also possible to consider re-parametrizations of critical points of the functional

$$L_{-1}(u) = L_{-1}([a,b];u) := \int_a^b \sqrt{(V_\varepsilon(u) - 1)\,|\dot{u}|^2}, \tag{24}$$

which is defined in the closure with respect to the weak topology of H^1 of

$$H_{-1} = H_{-1}^{p_1 p_2}([a,b]) := \{u \in H_{p_1 p_2}([a,b]) : V(u(t)) > 1, |\dot{u}(t)| > 0 \text{ for a.e. } t \in [a,b]\}\,.$$

Actually local minimizers of M_{-1} are local minimizers of L_{-1}, and the converse is true up to a re-parameterization. The functional L_{-1} has a useful geometric meaning, since for $u \in H_{-1}$ the value $L_{-1}(u)$ is the length of the curve parametrized by u with respect to the Jacobi metric $g_{ij}(y) = (V_\varepsilon(y) - 1)\,\delta_{ij}$, where δ_{ij} is the Kronecker's delta; this metric makes the set $\{V_\varepsilon(u) > 1\}$ a Riemannian manifold.

Let us look at Theorem 4.12 of [11]. We proved that there exists $\varepsilon_3 > 0$ such that for every $\varepsilon \in (0, \varepsilon_3)$, $p_1, p_2 \in \partial B_R(0)$ and $l \in \mathfrak{I}^N$ problem (20) has a solution $\mathfrak{y}_l(\cdot\,; p_1, p_2; \varepsilon) \in K_l^{p_1 p_2}([0,T])$ ($T = T(p_1, p_2; \varepsilon; l)$) which is a re-parametrization of a local minimizer of the Maupertuis' functional M_{-1} in $K_l^{p_1 p_2}([0,1])$, for some $T > 0$. If $p_1 = p_2$ and

$$l_1 = \cdots = l_{j-1} = l_{j+1} = \cdots = l_N \neq l_j \mod 2, \tag{25}$$

then this solution could be an ejection-collision solution with a unique collision in c_j, otherwise it has to be self-intersection-free and collision-free. The successive step consisted in the translation of Theorem 4.12 in the language of partitions. This is possible since if $u \in \widehat{K}_l$ is self-intersection-free then it separates the centres in two different groups, which are determined by the particular choice of $l \in \mathfrak{I}^N$; namely, a self-intersection-free path in a class $\widehat{K}_l$ induces a partition of the centres in two non-empty sets. Hence we could define the application $\mathcal{A} : \mathfrak{I}^N \to \mathcal{P}$ which associates to a winding vector

$$l = (l_1, \ldots, l_N) \text{ with } \begin{cases} l_k \equiv 0 \mod 2 & k \in A_0 \subset \{1, \ldots, N\} \\ l_k \equiv 1 \mod 2 & k \in A_1 \subset \{1, \ldots, N\} \end{cases}$$

the partition

$$\mathcal{A}(l) := \{\{c_k : l_k \in A_0\}, \{c_k : l_k \in A_1\}\}.$$

It is then natural to set

$$\begin{aligned} \widehat{K}_{P_j} &= \widehat{K}_{P_j}^{p_1 p_2}([a,b]) := \left\{u \in \widehat{K}_l^{p_1 p_2}([a,b]) : l \in \mathcal{A}^{-1}(P_j)\right\}, \\ K_{P_j} &= K_{P_j}^{p_1 p_2}([a,b]) := \left\{u \in K_l^{p_1 p_2}([a,b]) :\in \mathcal{A}^{-1}(P_j)\right\}. \end{aligned}$$

In comparison with [11], note that we don't require that a path in K_{P_j} has no self-intersection; for the N-centre problem such a requirement was proved to be natural, in the sense that every minimizer of the Maupertuis' functional in $\widehat{K}_l$ is necessarily self-intersection-free, unless it is an ejection-collision

minimizer; for the rotating problem this is not necessarily true, therefore we drop this condition in the definition of $\widehat{K}_{P_j}$.
From Theorem 4.12, we obtained, for every $\varepsilon \in (0, \varepsilon_3)$, $p_1, p_2 \in \partial B_R(0)$ and $P_j \in \mathcal{P}$, the existence of a solution $\mathfrak{y}_{P_j}(\cdot\,; p_1, p_2; \varepsilon)$ of problem (20), which is a re-parametrization of a local minimizer of the Maupertuis' functional M_{-1} in $K_{P_j}^{p_1 p_2}([0,1])$. If $p_1 = p_2$ and $P_j \in \mathcal{P}_1$ then $\mathfrak{y}_{P_j}(\cdot\,; p_1, p_2; \varepsilon)$ can be an ejection-collision solution with a unique collision in c_i, otherwise it is always collision-free (recall the definition of $\mathcal{P}_1$, equation (5)).

Let's come back to our "fixed Jacobi constant problem"

$$\begin{cases} \ddot{y}(t) + 2\nu' i \dot{y}(t) = \nabla \Phi_{\nu',\varepsilon}(y(t)) & t \in [0,T] \\ \frac{1}{2}|\dot{y}(t)|^2 - \Phi_{\nu',\varepsilon}(y(t)) = -1 & t \in [0,T] \\ |y(t)| < R & t \in [0,T] \\ y(0) = p_1 \qquad y(T) = p_2. \end{cases} \tag{26}$$

The variational formulation of (26) will be the object of subsection 4.1. We will state the main result of this section in subsection 4.2.

4.1 The variational formulation

Let us consider a general problem of type

$$\begin{cases} \ddot{z}(t) + 2\nu i \dot{z}(t) = \nabla \Phi_{\nu}(z(t)) & t \in [0,T] \\ \frac{1}{2}|\dot{z}(t)|^2 - \Phi_{\nu}(z(t)) = h & t \in [0,T] \\ z(0) = p_1 \qquad z(T) = p_2. \end{cases} \tag{27}$$

with $T > 0$ to be determined and $p_1, p_2 \in \mathbb{R}^2$. In order to solve it, we cannot use the Maupertuis' functional because it is suited for fixed energy problems. However, exploiting the existence of the Jacobi constant, we can study the Maupertuis-type functional

$$M_{h,\nu}([a,b]; u) := \sqrt{2} \left(\int_a^b |\dot{u}|^2 \right)^{\frac{1}{2}} \left(\int_a^b \Phi_\nu(u) + h \right)^{\frac{1}{2}} + \nu \int_a^b \langle iu, \dot{u} \rangle.$$

We will briefly write $M_{h,\nu}$ instead of $M_{h,\nu}([a,b];\cdot)$ when there is no possibility of misunderstanding. The domain of $M_{h,\nu}$ is the closure in the weak topology of H^1 of

$$H_{h,\nu}^{p_1 p_2}([a,b]) := \{ u \in H_{p_1 p_2}(a,b]) : \Phi_\nu(u(t)) > -h, |\dot{u}(t)| > 0 \text{ for a.e. } t \in [a,b] \}.$$

If

$$\sqrt{2} \left(\int_a^b |\dot{u}|^2 \right)^{\frac{1}{2}} \left(\int_a^b \Phi_\nu(u) + h \right)^{\frac{1}{2}} > 0, \tag{28}$$

we can set

$$\omega^2 := \frac{\int_a^b \Phi_\nu(u) + h}{\int_a^b |\dot{u}|^2} > 0 \tag{29}$$

and it makes sense to consider the re-parametrization $z(t) = u(\omega t)$, defined in $[a/\omega, b/\omega]$. The functional $M_{h,\nu}$ is differentiable over $\widehat{H} \cap \overline{H_{h,\nu}}^{\sigma(H^1,(H^1)^*)}$ (seen as an affine space on H_0^1). We will consider $[a,b] = [0,1]$ for the sake of simplicity.

Theorem 4.1. *Let $u \in \widehat{H}_{p_1 p_2}([0,1]) \cap \overline{(H^{p_1p_2}_{h,\nu}([0,1])}^{\sigma(H^1,(H^1)^*)}$ be a critical point of $M_{h,\nu}$, i.e. $dM_{h,\nu}(u)[v] = 0$ for every $v \in H^1_0([0,1])$, and assume that (28) is satisfied. Let ω be defined by (29).*
Then $z(t) := u(\omega t)$ is a classical solution of (27) with $T = 1/\omega$, while u itself is a classical solution of

$$\begin{cases} \omega^2\ddot{u}(t) + 2\nu\omega i\dot{u}(t) = \nabla\Phi_\nu(u(t)) & t \in [0,1], \\ \frac{1}{2}|\dot{u}(t)|^2 - \frac{\Phi(u(t))}{\omega^2} = \frac{h}{\omega^2} & t \in [0,1], \\ u(0) = p_1, \quad u(1) = p_2. \end{cases} \tag{30}$$

Proof. It is not difficult to check that if $dM_{h,\nu}(u)[v] = 0$ for every $v \in H^1_0([0,1])$ then $z(t) = u(\omega t)$ is a classical solution the first equation in (27). The Jacobi constant for z reads

$$\frac{1}{2}|\dot{z}(t)|^2 - \Phi_\nu(z(t)) = k \quad \forall t \quad \Leftrightarrow \quad \frac{\omega^2}{2}|\dot{u}(s)|^2 - \Phi_\nu(u(s)) = k \quad \forall s,$$

where $k \in \mathbb{R}$. We deduce

$$\omega^2 = \frac{\int_0^1 \Phi_\nu(u) + k}{\frac{1}{2}\int_0^1 |\dot{u}|^2};$$

comparing with (29), we obtain $k = h$. □

The previous statement says that the functional $M_{h,\nu}$ plays, for problem (27), the role that the classical Maupertuis' functional M_h plays for a fixed energy problem of type (20). In order to apply variational methods it is worthwhile working in H rather then in $\widehat{H}$, since $\widehat{H}$ is not weakly closed. As a consequence, it is not possible to rule out the occurrence of collisions from the beginning. This leads to the concept of weak solution for the problem (27).

Definition 2. Let u be a local minimizer of $M_{h,\nu}$ in $H^{p_1,p_2}_{h,\nu}([0,1])$ such that (28) holds true, and let ω be defined by (29). We say that $z(t) = u(\omega t)$ is a *weak solution* of (27) in the time interval $[0, 1/\omega]$.

If z is a weak solution, we can define the collision set as:

$$T_c(z) := \left\{ t \in \left[0, \frac{1}{\omega}\right] : z(t) = c_j \text{ for some } j = 1, \dots, N \right\}.$$

It is not difficult to check that if z is a weak solution and $(a,b) \subset [0,1] \setminus T_c(z)$, then z is a classical solution of the restricted problem in (a,b), with Jacobi constant h: indeed for every $\varphi \in \mathcal{C}^\infty_c(a,b)$ it results

$$\frac{d}{d\lambda} M_{h,\nu}(u + \lambda\varphi)\Big|_{\lambda=0} = 0. \tag{31}$$

One can verify that the set $T_c(z)$ is discrete and finite, so that z is a classical solution almost everywhere in $[0, 1/\omega]$. On the other hand, a local minimizer in K_l of $M_{h,\nu}$ does not satisfy the motion equation in every time interval $[c,d]$ such that $|u(t)| = R$ for every $t \in [c,d]$; indeed, in such a situation it is not true anymore that (31) holds true for every variation $\varphi \in \mathcal{C}^\infty_c([c,d])$. Nevertheless, the conservation of the Jacobi constant still holds true.

Proposition 4.2. *If $u \in \overline{(H^{p_1p_2}_{h,\nu}([0,1])}^{\sigma(H^1,(H^1)^*)}$ is a local minimizer of $M_{h,\nu}$, then*

$$\frac{1}{2}|\dot{u}(t)|^2 - \frac{\Phi_\nu(u(t))}{\omega^2} = \frac{h}{\omega^2} \qquad \textit{for a.e. } t \in [0,1]$$

Proof. It is a consequence of the extremality of u with respect to time re-parametrization keeping the ends fixed. For every $\varphi \in \mathcal{C}_c^\infty((0,1),\mathbb{R})$, let us consider $u_\lambda(t) := u(t+\lambda\varphi(t))$. For λ sufficiently small the function $t \mapsto t+\lambda\varphi(t)$ is increasing in $[0,1]$, so that in particular it is invertible; the minimality of u implies

$$\frac{d}{d\lambda} M_{h,\nu}(u_\lambda)\bigg|_{\lambda=0} = 0.$$

□

Remark 5. Note that, if $\nu = 0$, the functional $M_{h,\nu}$ reduces to

$$M_{h,0}(u) := \sqrt{2}\left(\int_a^b |\dot u|^2\right)^{\frac12}\left(\int_a^b (V(u)+h)\right)^{\frac12} = 2\sqrt{M_h(u)},$$

where M_h is the classical Maupertuis' functional of type (23). This reflects the perturbed nature of problem (26). Actually, due to the monotonicity of the square root for positive values of its argument it is immediate to deduce that u is a (local) minimizer of M_h at a positive level if and only if it is a (local) minimizer of $M_{h,0}$ such that (28) is satisfied. Therefore, if we work in a set in which M_h is bounded below by a positive constant, it is equivalent to minimize M_h or $M_{h,0}$. In particular, since in Lemma 4.16 of [11] we proved that for every $p_1, p_2 \in \partial B_R(0)$ and for every $l \in \mathfrak{I}^N$ there exists $C > 0$ such that

$$M_{-1}(u) \geq C > 0 \qquad \forall u \in K_l^{p_1p_2}([0,1]),$$

the characterization of the minimizers of M_{-1} in K_l (and consequently also in K_{P_j}) described in Theorem 4.12 of [11] (or Corollary 4.14 of [11]) applies for the minimizers of $M_{-1,0}$; this will be crucial in section 6.

As announced in section 1, there is an analogue counterpart for the functional L_h. We introduce $L_{h,\nu}([a,b];\cdot) : \overline{H_{h,\nu}}^{\sigma(H^1,(H^1)^*)} \to \mathbb{R}\cup\{+\infty\}$ as

$$L_{h,\nu}([a,b];u) := \int_a^b \sqrt{(\Phi_\nu(u)+h)}|\dot u| + \frac{1}{\sqrt{2}}\nu\int_a^b \langle iu, \dot u\rangle.$$

For $u \in H^1([a,b])$ let us consider the following class of orientation-preserving re-parametrizations

$$\Gamma_u := \left\{([c,d],f) : f : [c,d] \to [a,b], f \in \mathcal{C}^1([c,d],\mathbb{R}) \text{ and increasing, such that } u\circ f \in H^1([c,d])\right\}.$$

It is not difficult to check that $L_{h,\nu}$ is invariant under re-parametrizations of Γ_u. We point out that this is false if we consider re-parametrizations which don't preserve the orientation. In particular, differently from L_h, $L_{h,\nu}$ is not a length. It is possible to check that if $|\nu|$ is sufficiently small then

$$\sqrt{\Phi_\nu(z)+h}|\dot z| + \nu\langle iu, \dot u\rangle$$

is a Finsler function which makes the "Hill's region" $\{\Phi_\nu(z) > -h\}$ a Finsler manifold.

Theorem 4.3. *Let $u \in H_{h,\nu}^{p_1p_2}([0,1]) \cap \widehat{H}_{p_1p_2}([0,1])$ be a non-constant critical point of $L_{h,\nu}$. Then there exist a re-parametrization z of u which is a classical solution of (27) for some $T > 0$.*

Proof. We can adapt the proof of Theorem 4.5 of [11] with minor changes. □

The relationship between minimizers of $M_{h,\nu}$ and $L_{h,\nu}$ is given by the following statement.

Proposition 4.4. *Let $u \in H_{h,\nu} \cap \widehat{H}$ be a non-constant (local) minimizer of $M_{h,\nu}$ such that (28) holds true. Then u is a (local) minimizer of $L_{h,\nu}$ in $H_{h,\nu} \cap \widehat{H}$.*
On the other hand, let $u \in H_{h,\nu} \cap \widehat{H}$ be a non-constant (local) minimizer of $L_{h,\nu}$. Then, up to a re-parametrization, u is a (local) minimizer of $M_{h,\nu}$ in $H_{h,\nu} \cap \widehat{H}$ such that (28) holds true.

Proof. Due to the Hölder inequality we have

$$\sqrt{2} L_{h,\nu}(u) \leq M_{h,\nu}(u) \qquad \forall u \in H_{h,\nu} \cap \widehat{H},$$

with equality if and only if there exists $C > 0$ such that

$$|\dot{u}(t)|^2 = C\left(\Phi_\nu(u(t)) - 1\right) \qquad \forall t \in [0,1].$$

Now we can follow step by step the proofs of Proposition 4.6 and Proposition 4.7 of [11]. $\square$

4.2 Existence of inner solutions

The following result is a partial counterpart of Theorem 4.12 of [11].

Proposition 4.5. *There exist $\varepsilon_4 > 0$ and $\nu_2' > 0$ such that for every $(p_1, p_2, \varepsilon, \nu', l) \in (\partial B_R(0))^2 \times (0, \varepsilon_4) \times (-\nu_2', \nu_2') \times \mathfrak{I}^N$, problem (26) has a weak solution $y_l(\cdot\,; p_1, p_2; \varepsilon, \nu') \in K_l^{p_1 p_2}([0,T])$ which is a re-parametrization of a local minimizer $u_l(\cdot; p_1, p_2; \varepsilon, \nu')$ of the Maupertuis' functional $M_{-1,\nu'}$ in $K_l^{p_1 p_2}([0,1])$.*

Before proceeding with the proof of Theorem 4.5, we state the translation of this result in terms of partitions.

Corollary 4.6. *For every $(p_1, p_2, \varepsilon, \nu', P_j) \in (\partial B_R(0))^2 \times (0, \varepsilon_4) \times (-\nu_2', \nu_2') \times \mathcal{P}$, problem (26) has a weak solution $y_{P_j}(\cdot\,; p_1, p_2; \varepsilon, \nu') \in K_{P_j}^{p_1 p_2}([0,T])$ which is a re-parametrization of a local minimizer $u_{P_j}(\cdot; p_1, p_2; \varepsilon, \nu')$ of the Maupertuis'-type functional $M_{-1,\nu'}$ in $K_{P_j}^{p_1 p_2}([0,1])$.*

We fix $[a,b] = [0,1]$ and the Jacobi constant to -1, so we will write $M_{\nu'}$ instead of $M_{-1,\nu'}$. Also, we fix $p_1, p_2 \in \partial B_R(0)$ and $l \in \mathfrak{I}^N$.

Remark 6. In the statement of Theorem 4.5 the values ε_4 and ν_2' depend neither on $p_1, p_2 \in \partial B_R(0)$, nor on $l \in \mathfrak{I}^N$. But here we fixed p_1, p_2 and l before finding ε_4 and ν_2'. Actually, once we will find ε_4 and ν_2', we will see that they are independent on the previous quantities.

We aim at applying the direct methods of the calculus of variations in order to find a minimizer of $M_{\nu'}$ in K_l. Assuming that we can find such a minimizer $u_l(\cdot\,; p_1, p_2; \varepsilon, \nu')$, in order to obtain a weak solution of (26) we have to show that

$$1)\ u_l(\cdot\,; p_1, p_2; \varepsilon, \nu') \text{ satisfies (28)}, \qquad 2)\ |u_l(t; p_1, p_2; \varepsilon, \nu')| < R \quad \forall t \in (0,1).$$

Note that the first requirement is satisfied: for every $u \in \bigcup_{p_1, p_2, l} K_l^{p_1 p_2}([0,1])$, it results $|u| \leq R$; therefore we can use the bound of Remark 3. We will discuss about the second condition after the minimization.

Lemma 4.7. *The functional $M_{\nu'}$ is coercive in K_l.*

Proof. Let $(u_n) \subset K_l$ such that $\|\dot{u}_n\|_{H^1} \to \infty$ for $n \to \infty$. Since $\|u_n\|_{L^2} \leq R$, necessarily $\|\dot{u}_n\|_{L^2} \to +\infty$ as $n \to \infty$. As $V_\varepsilon(y) - 1 \geq M_1 > 0$ in $B_R(0)$,

$$\begin{aligned} M_{\nu'}(u_n) &\geq \sqrt{2}\|\dot{u}_n\|_{L^2} \left(M_1 + \frac{(\nu')^2}{2} \int_0^1 |u_n|^2 \right)^{\frac{1}{2}} - |\nu'| \int_0^1 |u_n||\dot{u}_n| \\ &= \sqrt{2}\|\dot{u}_n\|_{L^2} \left(\frac{|\nu'|}{\sqrt{2}} \|u_n\|_{L^2} + \lambda \right) - |\nu'| \|u_n\|_{L^2} \|\dot{u}_n\|_{L^2} \end{aligned}$$

for some $\lambda > 0$. Hence $M_{\nu'}(u_n) \geq \sqrt{2}\lambda \|\dot{u}_n\|_{L^2}$. ∎

Lemma 4.8. *The functional $M_{\nu'}$ is weakly lower semi-continuous in K_l.*

Proof. Let $(u_n) \subset K_l$ such that $u_n \rightharpoonup u$ weakly in H^1. It is by now standard the proof of

$$\left(\int_0^1 |\dot{u}| \right)^{\frac{1}{2}} \left(\int_0^1 \Phi_{\nu',\varepsilon}(u) - 1 \right)^{\frac{1}{2}} \leq \liminf_{n \to \infty} \left(\int_0^1 |\dot{u}_n| \right)^{\frac{1}{2}} \left(\int_0^1 \Phi_{\nu',\varepsilon}(u_n) - 1 \right)^{\frac{1}{2}},$$

see for instance [2, 13]. It remains to show that

$$\nu' \int_0^1 \langle iu, \dot{u} \rangle \leq \liminf_{n \to \infty} \nu' \int_0^1 \langle iu_n, \dot{u}_n \rangle. \tag{32}$$

The weak convergence of u_n to u implies that $u_n \to u$ uniformly in $[0,1]$ and $\dot{u}_n \rightharpoonup \dot{u}$ weakly in L^2, as $n \to \infty$. We have

$$\nu' \int_0^1 \langle iu_n, \dot{u}_n \rangle = \nu' \int_0^1 \langle i(u_n - u), \dot{u}_n \rangle + \nu' \int_0^1 \langle iu, \dot{u}_n \rangle.$$

The first term tends to 0 and the second term tends to $\nu' \int_0^1 \langle iu, \dot{u} \rangle$ as $n \to \infty$; (32) follows. ∎

Remark 7. The term $\nu \int_0^1 \langle iu, \dot{u} \rangle$ is not only weakly lower semi-continuous in H^1, but also continuous in the weak topology of H^1.

Due to the coercivity and the weak lower semi-continuity of $M_{\nu'}$, we can apply the direct methods of the calculus of variations on the functional $M_{\nu'}$ in the weakly closed set K_l. For every $(\varepsilon, \nu') \in (0, \varepsilon_1/2) \times \mathbb{R}$, we obtain a minimizer $u_l(\cdot; p_1, p_2; \varepsilon, \nu')$ for which (28) is satisfied. The following result concludes the proof of Proposition 4.5.

Lemma 4.9. *There are $\varepsilon_4, \nu_2' > 0$ such that for every $(p_1, p_2, \varepsilon, \nu', l) \in (\partial B_R(0))^2 \times (0, \varepsilon_4) \times (-\nu_2', \nu_2') \times \mathfrak{I}^N$ the minimizer $u_l(\cdot\,; p_1, p_2; \varepsilon, \nu')$ is such that*

$$|u_l(\cdot\,; p_1, p_2; \varepsilon, \nu')| < R \qquad \forall t \in (0,1).$$

Proof. We can follow the same line of reasoning which was used in [11] in order to prove Proposition 4.22. For the reader's convenience, we report here the ingredients of the proof. Let us term

$$T_R(u) := \{t \in [0,1] : |u(t)| = R\}, \quad T^+_{R/2}(u) := \left\{ t \in [0,1] : |u(t)| > \frac{R}{2} \right\}$$

A connected component of $T_R(u)$ is an interval (possibly a single point) $[t_1, t_2]$ with $t_1 \le t_2$. It is possible to show that $u \in \mathcal{C}^1([0,1])$, and if (a,b) is a connected component of $T^+_{R/2}(u) \setminus T_R(u)$, then $u|_{(a,b)}$ is of class $\mathcal{C}^2$ and is a solution of

$$\omega^2 \ddot{u}(t) + 2\nu' \omega i \dot{u}(t) = \nabla \Phi_{\nu',\varepsilon}(u(t)), \quad \text{where} \quad \omega^2 := \frac{\int_0^1 (\Phi_{\nu'\varepsilon}(u) - 1)}{\frac{1}{2}\int_0^1 |\dot{u}|^2}.$$

Moreover, there are $\varepsilon_4, \nu'_2, \tau > 0$ such that, if $(\varepsilon, \nu') \in (0, \varepsilon_4) \times (-\nu'_2, \nu'_2)$, then for every t_3, t_4 such that

$$|u(t_3)| = R, \quad |u(t_4)| = \frac{R}{2}, \quad \frac{R}{2} < |u(t)| < R \quad \forall t \in \begin{cases} (t_3, t_4) & \text{if } t_3 < t_4 \\ (t_4, t_3) & \text{if } t_3 > t_4 \end{cases},$$

there holds $|t_4 - t_3| \le \tau$. Neither ε_4 nor ν'_2 depend on p_1, p_2 or l. Let $[t_1, t_2]$ be a connected component of $T_R(u)$, let (a,b) be a connected component of $T^+_{R/2}$ such that $[t_1, t_2] \subset (a,b)$. Let us consider $y(t) := u(\omega t)$. Since $y \in \mathcal{C}^1((a/\omega, b/\omega))$, it can lean against the circle $\{y \in \mathbb{R}^2 : |y| = R\}$ with tangential velocity, and for every $\lambda > 0$ there exists $t_5 > t_2$ (or $t_5 < t_1$, and in this case the following inequality has to be changed in obvious way) such that

$$\left| y\left(\frac{t_5}{\omega}\right) - Re^{i\vartheta(t_2/\omega)} \right| + \left| \dot{y}\left(\frac{t_5}{\omega}\right) - R\dot{\vartheta}\left(\frac{t_2}{\omega}\right) ie^{i\vartheta(t_2/\omega)} \right| < \lambda.$$

Thus, recalling that R is the radius of the circular solution of energy -1 for the α-Kepler's problem, the theorem of continuous dependence of the solutions with respect to the vector field and the initial data implies that y cannot enter (or exit from) the ball $B_{R/2}(0)$ in time τ, provided ε_4 and ν'_2 are sufficiently small (if this was not true, we can replace them with smaller quantities); this is in contradiction with the choice of l. □

In order to exploit the description of the behavior of the solution which we obtained for the N-centre problem in Theorem 4.12 of [11], we will replace ε_4 with $\min\{\varepsilon_3, \varepsilon_4\}$ (for the reader's convenience, we recall again that ε_3 has been introduced in Theorem 4.12 of [11]).

Definition 3. Let us fix **arbitrarily** $\nu'_3 \in \left(0, \min\{\nu'_2, \sqrt{2M_1}/R\}\right)$. For every $\varepsilon \in (0, \varepsilon_4)$ we term

$$\mathcal{IM}_\varepsilon := \{u_l(\cdot; p_1, p_2; \varepsilon, \nu') : \; p_1, p_2 \in \partial B_R(0), \; l \in \mathbb{Z}_2^N, \; |\nu'| < \nu'_3\},$$

the set of the *inner minimizers* of $\{M_{\nu'}\}_{|\nu'| < \nu'_3}$ for a fixed value of ε, and

$$\mathcal{IS}_\varepsilon := \{y_l(\cdot; p_1, p_2; \varepsilon, \nu') : \; p_1, p_2 \in \partial B_R(0), \; l \in \mathbb{Z}_2^N, \; |\nu'| < \nu'_3\},$$

the set of the corresponding *inner solutions* for a fixed value of ε.

We conclude this section with a collection of boundedness properties for the functions of $\mathcal{IM}_\varepsilon$.

Proposition 4.10. *Let $\varepsilon \in (0, \varepsilon_4)$. There are $C_1, C_2, C_3, C_4, C_5 > 0$ such that*

$$C_1 \le \inf_{u \in \mathcal{IM}_\varepsilon} \|\dot{u}\|_{L^2} \le \sup_{u \in \mathcal{IM}_\varepsilon} \|\dot{u}\|_{L^2} \le C_2,$$

$$C_3 \le \inf_{u = u_l(\cdot\,; p_1, p_2, \varepsilon, \nu') \in \mathcal{IM}_\varepsilon} \int_0^1 \Phi_{\nu',\varepsilon}(u) - 1 \le \sup_{u = u_l(\cdot\,; p_1, p_2, \varepsilon, \nu') \in \mathcal{IM}_\varepsilon} \int_0^1 \Phi_{\nu',\varepsilon}(u) - 1 \le C_4,$$

$$\sup_{u = u_l(\cdot\,; p_1, p_2, \varepsilon, \nu')} M_{\nu'}(u) \le C_5.$$

Remark 8. Since $\sup\{\|u\|_{L^2} : u \in \mathcal{IM}_\varepsilon \leq R\}$, the set $\mathcal{IM}_\varepsilon$ is bounded in the H^1 norm.

Proof. Every $u \in \mathcal{IM}_\varepsilon$ is of type $u_l(\cdot\,; p_1, p_2; \varepsilon, \nu')$ for some $p_1, p_2 \in \partial B_R(0)$, $l \in \mathfrak{I}^N$, $\nu' \in (\nu_3', \nu_3')$. Since $\mathfrak{I}^N$ is discrete and finite, we can prove the statement for a fixed l. In [11] we proved that the functions of $\bigcup_{p_1,p_2\in\partial B_R(0)} K_l^{p_1 p_2}([0,1])$ are uniformly non-constant, which ensures the existence of C_1. Furthermore, as an immediate consequence of the estimate in Remark 3, we obtain $C_3 = M_1$. Now let us fix $\widetilde{p}_1, \widetilde{p}_2 \in \partial B_R(0)$; there exists $\widetilde{u} \in K_l^{\widetilde{p}_1\widetilde{p}_2}([0,1])$ such that, for some $C_6 > 0$ and $\mu = \mu(\varepsilon) \in (0, \varepsilon)$, it results

$$|\dot{\widetilde{u}}(t)| = C_6, \quad |\widetilde{u}(t) - c_j| \geq \mu(\varepsilon) \quad \forall t \in [0,1], \forall j \in \{1, \dots, N\}.$$

For every $\nu' \in (-\nu_3', \nu_3')$ we have

$$\int_0^1 \Phi_{\nu',\varepsilon}(\widetilde{u}) = \int_0^1 \left(V_\varepsilon(\widetilde{u}) + \frac{(\nu')^2}{2}|\widetilde{u}|^2 \right) \leq \frac{M}{\alpha\mu^\alpha} + \frac{(\nu_3')^2}{2}R^2 =: C_7,$$

where $C_7 = C_7(\varepsilon)$. Starting from this bound it is possible to obtain a uniform bound with respect to p_1, p_2, ν' for the level of the minimizers of $M_{\nu'}$. If $(p_1, p_2) \neq (\widetilde{p}_1, \widetilde{p}_2)$, we consider the path

$$\widehat{u}(t) := \begin{cases} \zeta_R(3t; p_1, \widetilde{p}_1) & t \in [0, 1/3] \\ \widetilde{u}(3t-1) & t \in (1/3, 2/3] \\ \zeta_R\left(3t-2; \widetilde{p}_2, p_2\right) & t \in (2/3, 1], \end{cases}$$

where, for $p_*, p_{**} \in \partial B_R(0)$, $\zeta_R(\cdot\,; p_*, p_{**}) : [0,1] \to \mathbb{R}^2$ parametrizes the shorter (in the Euclidean metric) arc of $\partial B_R(0)$ connecting p_* and p_{**} with constant velocity. As far as the velocity of $\zeta_R(\cdot\,; p_*, p_{**})$ is concerned, it is easy to see that it is uniformly bounded with respect to p_*, p_{**}. This, together with the assumptions on $\widetilde{u}$, implies that also the velocity of $\widehat{u}$ is bounded in $[0,1]$, and

$$M_{\nu'}(\widehat{u}) \leq C \left(\int_0^1 \Phi_{\nu',\varepsilon}(\widetilde{u}) - 1 + C \right)^{\frac{1}{2}} + |\nu'|RC \leq C_5.$$

Again, $C_5 = C_5(\varepsilon) > 0$, while it does not depend on the ends p_1 and p_2 or on the parameter ν'. Consequently, for the family of the minimizers there holds

$$M_\nu'(u_l(\cdot\,; p_1, p_2; \varepsilon, \nu')) \leq C_5 \qquad \forall p_1, p_2 \in \partial B_R(0),\ |\nu'| < \nu_3'. \tag{33}$$

Using (14), we obtain

$$\begin{aligned} \|\dot{u}_l(\cdot\,; p_1, p_2; \varepsilon, \nu')\|_{L^2} &\leq \frac{C_5 - \nu' \int_0^1 \langle i u_l(\cdot\,; p_1, p_2; \varepsilon, \nu'), \dot{u}_l(\cdot\,; p_1, p_2; \varepsilon, \nu')\rangle}{\sqrt{2M_1}} \\ &\leq \frac{C_5 + |\nu'|R\|\dot{u}_l(\cdot\,; p_1, p_2; \varepsilon, \nu')\|_{L^2}}{\sqrt{2M_1}}, \end{aligned}$$

for every $p_1, p_2 \in \partial B_R(0)$ and $|\nu'| < \nu_3'$. Now

$$\left(1 - \frac{|\nu'|R}{\sqrt{2M_1}}\right) \|\dot{u}_l(\cdot\,; p_1, p_2; \varepsilon, \nu')\|_{L^2} \leq \frac{C_5}{\sqrt{2M_1}}.$$

Since $|\nu'| < \nu_3' < \sqrt{2M_1}/R$, the coefficient on the left hand side is bounded below by a positive constant; therefore

$$\|\dot{u}_l(\cdot\,; p_1, p_2; \varepsilon, \nu')\|_{L^2} \leq \frac{C_5}{\sqrt{2M_1}} \left(1 - \frac{|\nu_3'|R}{\sqrt{2M_1}}\right)^{-1} =: C_2(\varepsilon) \qquad \forall (p_1, p_2, \nu') \in (\partial B_R(0))^2 \times (-\nu_3', \nu_3').$$

It remains to find C_4; from (33), using the existence of C_1, it follows

$$\left(\int_0^1 \Phi_{\nu',\varepsilon}(u_l(\cdot\,;p_1,p_2;\varepsilon,\nu'))-1\right)^{\frac{1}{2}} \le \frac{C_5+|\nu'|R\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|_{L_2}}{\sqrt{2}\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|_{L^2}} \le \frac{C_5}{\sqrt{2}C_1}+\frac{\nu_3' R}{\sqrt{2}} =: C_4^{\frac{1}{2}}. \quad \square$$

Remark 9. The fact that some constants depend on ε reflects the fact that more the Jacobi constant is small, more the admissible values of the angular velocity are small, see Remark 2. This is why we keep ε fixed, letting ν' vary, instead of considering both ε and ν' as parameters.

We termed $[0, T_l(p_1,p_2;\varepsilon,\nu')]$ as the time interval of $y_l(\cdot\,;p_1,p_2;\varepsilon,\nu') \in \mathcal{IS}_\varepsilon$. It results

$$T_l(p_1,p_2;\varepsilon,\nu') = \frac{1}{\omega_l(p_1,p_2;\varepsilon,\nu')}, \quad \text{where} \quad \omega_l(p_1,p_2;\varepsilon,\nu') = \frac{\int_0^1 \Phi_{\nu',\varepsilon}(u_l(\cdot\,;p_1,p_2;\varepsilon,\nu'))-1}{\frac{1}{2}\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|^2}.$$

Corollary 4.11. *Let $\varepsilon \in (0,\varepsilon_4)$. There exist $C_1, C_2, C_3 > 0$ such that*

$$C_1 \le T_l(p_1,p_2;\varepsilon,\nu') \le C_2$$
$$\|y_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|_{H^1([T_l(p_1,p_2;\varepsilon,\nu')])} \le C_3$$

for every $(p_0,p_1,\nu',l) \in (\partial B_R(0))^2 \times (-\nu_3',\nu_3') \times \mathfrak{I}^N$.

4.3 Forward normal neighborhoods

In [11], we exploited the geometric interpretation of L: it is the length in the Riemannian manifold $\{V_\varepsilon(y) > -1\}$ endowed with the Jacobi metric. In particular in section 5 of the quoted paper we used classical results concerning the existence of totally normal and strongly convex neighborhoods (for the definitions, see [5]). Now we are not dealing with a length anymore, but with a Finsler function; so, something similar can be proven. The following is a known result, but since we cannot find a proper reference we give a sketch of the proof for completeness.

Proposition 4.12. *Let $\rho > 0$ be small enough, in such a way that $B_\varepsilon(0) \subset \overline{B_{R/2-\rho}(0)} \subset \overline{B_{R+\rho}(0)} \subset \{\Phi_{\nu',\varepsilon}(y) > 1\}$ and $R/2-\rho > \varepsilon$. There exist $\varepsilon_5 \in (0,\varepsilon_4]$, $\nu_4' \in (0,\nu_3']$ and $\bar r \in (0,2\rho)$ such that if $\varepsilon \in (0,\varepsilon_5)$, $|\nu'| < \nu_4'$, $p_1,p_2 \in \overline{B_R(0)\setminus B_{R/2}(0)}$ and $|p_1-p_2| \le \bar r$ then there is a unique minimizer $u_{min}(\cdot\,;p_1,p_2;\varepsilon,\nu')$ of $M_{\nu'}$ in the set*

$$\left\{u \in H_{p_1p_2}([0,1]) : u(t) \in B_{R+\rho}(0)\setminus B_{R/2-\rho}(0)\ \forall t\right\}.$$

Moreover, it depends in a $\mathcal{C}^1$ way on its ends and on the parameters ε and ν', and is the unique global minimizer of $M_{\nu'}$ in $H_{p_1p_2}([0,1])$.

Definition 4. Let $\varepsilon \in (0,\varepsilon_5)$, $|\nu'| < \nu_4'$, and let us take $\rho > 0$ as above; let $p \in \overline{B_R(0)\setminus B_{R/2}(0)}$. For every pair $p_1,p_2 \in \overline{B_{\bar r/2}}(p)$ there is a unique (up to a re-parametrization) local minimizer of $L_{\nu'}$ which starts from p_1 and arrives at p_2, depending smoothly on the ends. We will say that $B_{\bar r/2}(p)$ is a *forward normal neighborhood* of p.

Proposition 4.12 says that every point of $\overline{B_R(0)\setminus B_{R/2}(0)}$ has a forward normal neighborhood; moreover, the set $B_{R+\rho}(0)\setminus B_{R/2-\rho}(0)$ is "convex", in the sense that the minimizers $u_{\min}(\cdot\,;p_1,p_2;\varepsilon,\nu')$ stay in it.

Forward normal neighborhoods plays the role of totally normal ones of a Riemaniann manifold,

with the difference that, since our functional $L_{\nu'}$ is not invariant under orientation-reversing reparameterizations, a minimizer of $L_{\nu'}$ in $H_{p_1p_2}([0,1])$ could not be a minimizer of $L_{\nu'}$ in $H_{p_2p_1}([0,1])$. Actually for every $p \in \{\Phi_{\nu',\varepsilon}(y) > 1\}$ it is possible to prove the existence of a forward normal neighborhood, but due to the degeneracy of our Finsler function, which can become even negative if we are close to the boundary of the "Hill's region", the radius of these neighborhood becomes smaller and smaller and tends to 0 as p approaches $\{\Phi_{\nu',\varepsilon}(y) = 1\}$.

Proof. Let $p_1, p_2 \in \overline{B_R(0) \setminus B_{R/2}(0)}$, $\varepsilon \in (0, \varepsilon_4)$, $\nu' \in (-\nu'_4, \nu'_4)$. The existence can be proved applying the direct methods of the calculus of variations. If $p_1 = p_2$, observe that the minimizer is simply the constant function p_1.
Let $u_{\min}(\cdot\,; p_1, p_2; \varepsilon, \nu')$ be a minimizer in $H_{p_1p_2}([0,1])$; there exists $\bar{r} > 0$ such that if $|p_1 - p_2| \le \bar{r}$, then $u_{\min}(\cdot\,; p_1, p_2; \varepsilon, \nu')$ is contained in $B_{R+\rho}(0) \setminus B_{R/2-\rho}(0)$: if not, there are sequences $(r_n) \subset \mathbb{R}^+$ and $((p_1^n, p_2^n)) \subset \overline{B_R(0) \setminus B_{R/2}(0)}$ such that $|p_1^n - p_2^n| \le r_n$ and $u_{\min}(\cdot\,; p_1^n, p_2^n; \varepsilon, \nu')$ touches $\partial\left(B_{R+\rho}(0) \setminus B_{R/2-\rho}(0)\right)$. But this is absurd, because if $r_n \to 0$ the minimizers tends to be constant functions in $\overline{B_R(0) \setminus B_{R/2}(0)}$. The value ρ is independent on $\varepsilon \in (0, \varepsilon_4)$ and $|\nu'| < \nu'_4$. For the uniqueness and the $\mathcal{C}^1$ dependence, we consider the map

$$\begin{aligned}\left(\overline{B_R(0) \setminus B_{R/2}(0)}\right)^2 \times (0, \varepsilon_4) \times (-\nu'_4, \nu'_4) \times H_{p_1p_2}([0,1]) &\to (H_{p_1p_2}([0,1]))^* \\ (p_1, p_2, \varepsilon, \nu', u) &\mapsto dM_{\nu'}(u).\end{aligned}$$

Let $\bar{u}$ be a minimizer of $M_{\nu'}$ in $H_{p_1p_2}([0,1])$, whose image is contained in $B_{R+\rho}(0) \setminus B_{R/2-\rho}(0)$; an explicit computation shows that, if $|p_1 - p_2|$ and ν' are sufficiently small, the second differential $d^2M_{\nu'}(u)$ is positive definite, so that it is invertible. Thus, the Implicit Function Theorem applies to give uniqueness and smooth dependence. ☐

Remark 10. In section 3 we prove that, if $p_1, p_2 \in \partial B_R(0)$ are sufficiently close together, we can find a "close to brake" solution of problem 15 which, of course, passes close to the boundary of the "Hill's region" $\{\Phi_{\nu',\varepsilon}(y) > 1\}$. This is not in contradiction with the previous result, since an outer solution parametrizes a non-minimal critical point of $L_{\nu'}$.

5 A finite-dimensional reduction

In this section we glue the fixed ends trajectories previously obtained, alternating outer and inner arcs in order to construct periodic orbits of the restricted problem (3) in the whole plane. Since in this procedure we need smooth junctions, we are going to use a variational argument which is essentially the same we introduced in [11]. Let us set $\widetilde{\varepsilon} := \min\{\varepsilon_2, \varepsilon_5\}$, $\widetilde{\nu}' := \min\{\nu'_1, \nu'_4\}$. The quantities ε_2 and ν'_1 have been introduced in Proposition 3.1 (recall also the definition of δ therein), while ε_5 and ν'_4 have been introduced in Proposition 4.12, respectively. This is the main result of this section.

Proposition 5.1. *There exist $\bar{\varepsilon}, \bar{\nu}' > 0$ such that for every $(\varepsilon, \nu') \in (0, \bar{\varepsilon}) \times (-\bar{\nu}', \bar{\nu}')$, for every $n \in \mathbb{N}$ and $(P_{j_1}, \dots, P_{j_n}) \in \mathcal{P}^n$ there exists a periodic weak solution $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ of problem (8), which depends on $(P_{j_1}, \dots, P_{j_n})$ in the following way: the image of $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ crosses $2n$ times within one period the circle $\partial B_R(0)$, at times $(t_k)_{k=0,\dots,2n-1}$, and*

- *in (t_{2k}, t_{2k+1}) the solution stays outside $B_R(0)$ and*

$$|\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}(t_{2k}) - \gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}(t_{2k+1})| < \delta.$$

- *in (t_{2k+1}, t_{2k+2}) the solution lies inside $B_R(0)$, and, if it does not collide against any centre, then it separates them according to the partition P_{j_k}.*

Let us fix $\varepsilon \in (0, \widetilde{\varepsilon})$, $|\nu'| < \widetilde{\nu}'$, $n \in \mathbb{N}$, $(P_{k_1}, P_{k_2}, \ldots, P_{k_n}) \in \mathcal{P}^n$. We define

$$D = \left\{ (p_0, \ldots, p_{2n}) \in (\partial B_R(0))^{2n+1} : |p_{2j+1} - p_{2j}| \le \delta \text{ for } j = 0, \ldots, n-1,\ p_{2n} = p_0 \right\}.$$

Let $(p_0, \ldots, p_{2n}) \in D$. For every $j \in \{0, \ldots, n-1\}$, we can apply Proposition 3.1 to obtain an outer solution $y_{2j}(t) := y_{\text{ext}}(t; p_{2j}, p_{2j+1}; \varepsilon, \nu')$ defined in $[0, T_{2j}]$, where $T_{2j} := T_{\text{ext}}(p_{2j}, p_{2j+1}; \varepsilon, \nu')$. We recall that y_{2j} depends on p_{2j} and p_{2j+1} in a $\mathcal{C}^1$ manner. Also, from Corollary 4.6 we obtain an inner weak solution $y_{2j+1}(t) := y_{P_{k_{j+1}}}(t; p_{2j+1}, p_{2j+2}; \varepsilon, \nu')$ defined in $[0, T_{2j+1}]$, where $T_{2j+1} := T_{P_{k_{j+1}}}(p_{2j+1}, p_{2j+2}; \varepsilon, \nu')$ (recall that $\nu'_4 < \nu'_3$). Being $L_{\nu'}$ invariant under orientation-preserving reparameterizations, y_{2j+1} is a local minimizer of the functional $L_{\nu'}([0, T_{2j+1}]; \cdot)$. We point out that y_{2j+1} could not be unique; however, if there is more then one minimizer of $L_{\nu'}$ in K_{P_j}, we can arbitrarily choose one of them.
We set $\mathfrak{T}_k := \sum_{j=0}^{k} T_j$, $k = 0, \ldots, 2n-1$, and

$$\gamma^{((P_{k_1}, \ldots, P_{k_n}), \varepsilon, \nu')}_{(p_0, \ldots, p_{2n})}(s) := \begin{cases} y_0(s) & s \in [0, \mathfrak{T}_0] \\ y_1(s - \mathfrak{T}_0) & s \in [\mathfrak{T}_0, \mathfrak{T}_1] \\ \vdots & \\ y_{2n-2}(s - \mathfrak{T}_{2n-3}) & s \in [\mathfrak{T}_{2n-3}, \mathfrak{T}_{2n-2}] \\ y_{2n-1}(s - \mathfrak{T}_{2n-2}) & s \in [\mathfrak{T}_{2n-2}, \mathfrak{T}_{2n-1}]. \end{cases} \tag{34}$$

The function $\gamma^{((P_{k_1}, \ldots, P_{k_n}), \varepsilon, \nu')}_{(p_0, \ldots, p_{2n})}$ is a piecewise differentiable $\mathfrak{T}_{2n-1}$-periodic function. It is a weak solution of the restricted problem (3) with Jacobi constant -1 in $[0, \mathfrak{T}_{2n-1}] \setminus \{0, \mathfrak{T}_0, \ldots, \mathfrak{T}_{2n-1}\}$, but in general is not $\mathcal{C}^1$ in $\{0, \mathfrak{T}_0, \ldots, \mathfrak{T}_{2n-1}\}$; however, the right and left limits of the derivatives in these points are finite, so that it is in H^1. It is also possible that $\gamma^{((P_{k_1}, \ldots, P_{k_n}), \varepsilon, \nu')}_{(p_0, \ldots, p_{2n})}$ has collisions. Thanks to Lemma 3.4 and Corollary 4.11, we are sure that the time interval of $\gamma^{((P_{k_1}, \ldots, P_{k_n}), \varepsilon, \nu')}_{(p_0, \ldots, p_{2n})}$ is bounded above and bounded below, uniformly with respect to $(p_0, \ldots, p_{2n}) \in D$, by positive constants; therefore for every $(p_0, \ldots, p_{2n}) \in D$ the period of the associated function is neither trivial, nor infinite.

We introduce a function $F = F_{((P_{k_1}, \ldots, P_{k_n}), \varepsilon, \nu')} : D \to \mathbb{R}$ defined by

$$F(p_0, \ldots, p_{2n}) := L_{\nu'}\left([0, \mathfrak{T}_{2n-1}]; \gamma^{((P_{k_1}, \ldots, P_{k_n}), \varepsilon, \nu')}_{(p_0, \ldots, p_{2n})}\right) = \sum_{j=0}^{2n-1} L_{\nu'}([0, T_j]; y_j).$$

Proposition 5.2. *There exists $(\bar{p}_0, \ldots, \bar{p}_{2n}) \in D$ which minimizes F. There exist $\bar{\varepsilon}, \bar{\nu}' > 0$ such that, for every $(\varepsilon, \nu') \in (0, \bar{\varepsilon}) \times (-\bar{\nu}', \bar{\nu}')$, the associated function $\gamma^{((P_{k_1}, \ldots, P_{k_n}), \varepsilon, \nu')}_{(p_0, \ldots, p_{2n})}$ is a periodic weak solution of the restricted problem (8). The values $\bar{\varepsilon}$ and $\bar{\nu}'$ depends neither on n, nor on $(P_{k_1}, \ldots, P_{k_n}) \in \mathcal{P}^n$.*

Remark 11. Proposition 5.1 is an immediate consequence of this statement.
From now on, we will write $\gamma^{((P_{k_1}, \ldots, P_{k_n}), \varepsilon, \nu')}$ to denote the periodic weak solution associated to an arbitrarily chosen minimizer of $F_{((P_{k_1}, \ldots, P_{k_n}), \varepsilon, \nu')}$.

We will reach the result through a series of Lemmas. We will follow the same sketch already used in [11], see also [12].

Lemma 5.3. *The function F is continuous, so that there exists a minimizer of F in the compact set D.*

Proof. Repeat the proof of step 1) of Theorem 5.3 of [11]. $\square$

Let $(\bar{p}_0, \dots, \bar{p}_{2n})$ be a minimizer of F. We aim at showing that the minimality of $(\bar{p}_0, \dots, \bar{p}_{2n})$ implies smoothness in the junction times for the associated periodic function $\gamma^{((P_{k_1},\dots,P_{k_n}),\varepsilon,\nu')}_{(\bar{p}_0,\dots,\bar{p}_{2n})}$. In order to prove it, we would like to write explicitly the equation $\nabla F(\bar{p}_0, \dots, \bar{p}_{2n}) = 0$. As we noticed in [12], it is not evident that this can be done, because of the lack of uniqueness of inner minimizers of $M_{\nu'}$ in K_{P_j}: for this reason it is not immediate that an inner solution depends smoothly on its ends. In order to overcome the problem, we can use Proposition 4.12: for any $j \in \{0, \dots, n-1\}$, we consider a forward normal neighborhood U_{2j+1} of the point $\bar{p}_{2j+1}$. Let us choose $t_* \in (0, T_{2j+1})$ such that

$$\widetilde{p}_{2j+1} := y_{2j+1}(t_*) \in U_{2j+1}, \quad |\widetilde{p}_{2j+1}| < R, \quad y\left([0, t_*]\right) \subset \left(B_R(0) \setminus B_{R/2}(0)\right);$$

There exists a unique minimizer $\widehat{y}(\cdot; \bar{p}_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon, \nu')$ of $M_{\nu'}$, and hence also of $L_{\nu'}$ (up to a re-parameterization), which connects p_{2j+1} and $\widetilde{p}_{2j+1}$ in time 1, and depends smoothly on its ends. For the uniqueness, $\widehat{y}$ has to be a re-parametrization of y_{2j+1}. Note that if $p_{2j+1} \in \overline{U_{2j+1} \cap B_R(0)}$, then there is a unique minimizer $\widehat{y}(\cdot; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon, \nu')$ of $M_{\nu'}$ which connects p_{2j+1} and $\widetilde{p}_{2j+1}$. We will consider its re-parametrization $\widetilde{y}(\cdot\,; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon)$ such that

$$\begin{cases} \ddot{\widetilde{y}}(t) + 2\nu' i \dot{\widetilde{y}}(t) = \nabla \Phi_{\nu',\varepsilon}(\widetilde{y}(t)) \\ \frac{1}{2}|\dot{\widetilde{y}}(t)|^2 - \Phi_{\nu',\varepsilon}(\widetilde{y}(t)) = -1, \end{cases}$$

denoting by $[0, T(p_{2j+1}, \widetilde{p}_{2j+1})]$ its domain. Due to the minimality of $\widehat{y}(\cdot\,; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon, \nu')$ for $L_{\nu'}$, such a re-parametrization exists, see Theorem 4.3. In this way

$$\widetilde{y}(\cdot\,; \bar{p}_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon, \nu') \equiv y_{P_{k_{j+1}}}(\cdot\,; \bar{p}_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon, \nu')|_{[0, T(\bar{p}_{2j+1}, \widetilde{p}_{2j+1})]}. \tag{35}$$

Let $D_{2j+1} := \{p_{2j+1} \in \left(\partial B_R(0) \cap \bar{U}_{2j+1}\right) : |\bar{p}_{2j} - p_{2j+1}| \le \delta\}$. We define $G_{2j+1} : D_{2j+1} \to \mathbb{R}$ by

$$\begin{aligned} G_{2j+1}(p_{2j+1}) := {} & L\left([0, T(p_{2j+1})]; y_{\text{ext}}(\cdot\,; \bar{p}_{2j}, p_{2j+1}; \varepsilon, \nu')\right) \\ & + L\left([0, T(p_{2j+1}, \widetilde{p}_{2j+1})]; \widetilde{y}(\cdot\,; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon, \nu')\right), \end{aligned}$$

where $T(p_{2j+1})$ denotes $T_{\text{ext}}(\bar{p}_{2j}, p_{2j+1}; \varepsilon, \nu')$ (we will adopt this notation in this section). Of course, with minor changes we can also define a function G_{2j}, for every $j \in \{0, \dots, 2n\}$. Note that G_k is continuous (for every k), since it is a sum of terms which are both continuous with respect to p_k. As a consequence, G_k has a minimum.

Lemma 5.4. *If $(\bar{p}_0, \dots, \bar{p}_{2n})$ is a minimizer for F, then $\bar{p}_k$ is a minimizer for G_k.*

Proof. The proof is the same of Lemma 1 of [12]. $\square$

The main reason to pass from the study of F to the study of the functions G_k is that, in contrast with F, G_k is differentiable for every k: let's think at $k = 2j + 1$; $L\left([0, T(p_{2j+1})]; y_{\text{ext}}(\cdot\,; \bar{p}_{2j}, p_{2j+1}; \varepsilon, \nu')\right)$ depends smoothly on p_{2j+1} for the differentiable dependence of outer solutions with respect to the ends, and $L\left([0, T(p_{2j+1}, \widetilde{p}_{2j})]; \widetilde{y}(\cdot\,; p_{2j+1}, \widetilde{p}; \varepsilon, \nu')\right)$ depends smoothly on p_{2j+1} for Proposition 4.12. Therefore the minimality of $\bar{p}_{2j+1}$ implies that

$$\text{if } \bar{p}_{2j+1} \in D^{\circ}_{2j+1} \quad \Rightarrow \quad \frac{\partial G_{2j+1}}{\partial p_{2j+1}}(\bar{p}_{2j+1}) = 0$$

(the notation D°_{2j+1} denotes the inner of D_{2j+1}). This partial derivative is a linear operator from the tangent space $T_{\bar p_{2j+1}}(\partial B_R(0))$ into $\mathbb{R}$. In what follows we will show that, if ε and ν' are small enough, $\bar p_k \in D^\circ_k$ for every k, and that the stationarity conditions are nothing but regularity conditions for the functions

$$\zeta_{2j}(t) := \begin{cases} y_{P_{k_{j-1}}}(t + T_{2j-1} - T(\widetilde{p}_{2j}, \bar p_{2j}); \bar p_{2j-1}, \bar p_{2j}; \varepsilon, \nu') & \text{if } t \in [0, T(\widetilde{p}_{2j}, \bar p_{2j})] \\ y_{\mathrm{ext}}(t - T(\widetilde{p}_{2j}, \bar p_{2j}); \bar p_{2j}, \bar p_{2j+1}; \varepsilon, \nu') & \text{if } t \in [T(\widetilde{p}_{2j}, \bar p_{2j}), T(\widetilde{p}_{2j}, \bar p_{2j}) + T(\bar p_{2j+1})] \end{cases}$$

and

$$\zeta_{2j+1}(t) := \begin{cases} y_{\mathrm{ext}}(t; \bar p_{2j}, \bar p_{2j+1}; \varepsilon, \nu') & \text{if } t \in [0, T(\bar p_{2j+1})] \\ y_{P_{k_{j+1}}}(t - T(\bar p_{2j+1}); \bar p_{2j}, \bar p_{2j+1}; \varepsilon, \nu') & \text{if } t \in [T(\bar p_{2j+1}), T(\bar p_{2j+1}) + T(\bar p_{2j+1}, \widetilde{p}_{2j+1})]. \end{cases}$$

Taking into account that ζ_k is (up to a time translation) the restriction of $\gamma^{((P_{k_1},\dots,P_{k_n}),\varepsilon,\nu')}$ on a neighbourhood of the junction time $\mathfrak{T}_{k-1}$, we obtain $\mathcal{C}^1$ regularity for $\gamma^{((P_{k_1},\dots,P_{k_n}),\varepsilon,\nu')}$ itself.

Lemma 5.5. *For every $j = 0, \dots, n-1$, $p_{2j} \in D_{2j}$, and for every $\varphi \in T_{p_{2j}}(B_R(0))$, we have*

$$\frac{\partial G_{2j}}{\partial p_{2j}}(p_{2j})[\varphi] = \frac{1}{\sqrt{2}} \langle \dot{\widetilde{y}}(T(\widetilde{p}_{2j}, p_{2j}); \widetilde{p}_{2j}, p_{2j}; \varepsilon, \nu') - \dot y_{ext}(0; p_{2j}, \bar p_{2j+1}; \varepsilon, \nu'), \varphi \rangle.$$

For every $j = 0, \dots, n-1$, $p_{2j+1} \in D_{2j+1}$, and for every $\varphi \in T_{p_{2j+1}}(B_R(0))$, we have

$$\frac{\partial G_{2j+1}}{\partial p_{2j+1}}(p_{2j+1})[\varphi] = \frac{1}{\sqrt{2}} \langle \dot y_{ext}(T(p_{2j+1}); \bar p_{2j}, p_{2j+1}; \varepsilon, \nu') - \dot{\widetilde{y}}(0; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon, \nu'), \varphi \rangle.$$

Proof. It is not restrictive to consider the derivative of G_1 to ease the notation. The same calculations work for the other cases. There holds

$$\frac{\partial G_1}{\partial p_1}(p_1) = \frac{\partial}{\partial p_1} L_{\nu'}\left([0, T(p_1)]; y_{\mathrm{ext}}(\cdot\,; \bar p_0, p_1; \varepsilon, \nu')\right) + \frac{\partial}{\partial p_1} L_{\nu'}\left([0, T(p_1, \widetilde{p}_1)]; \widetilde{y}(\cdot\,; p_1, \widetilde{p}_1; \varepsilon, \nu')\right). \quad (36)$$

Let us consider the first term in the right side, writing simply y_0 instead of $y_{\mathrm{ext}}(\cdot\,; \bar p_0, p_1; \varepsilon, \nu')$; we consider $u_0(t) = y_0(T_0 t)$, defined in $[0,1]$. It results

$$\begin{aligned} \frac{\partial}{\partial p_1} L_{\nu'}([0, T(p_1)]; y_0) &= \frac{\partial}{\partial p_1} L_{\nu'}([0,1]; u_0) \\ &= \frac{1}{\sqrt{2}} \int_0^1 \left[\langle \frac{\dot u_0}{T_0}, \frac{d}{dt} \frac{\partial u_0}{\partial p_1} \rangle + \langle T_0 \nabla \Phi_{\nu',\varepsilon}(u_0), \frac{\partial u_0}{\partial p_1} \rangle \right] + \frac{1}{\sqrt{2}} \nu' \int_0^1 \left(\langle i \frac{\partial u_0}{\partial p_1}, \dot u_0 \rangle + \langle i u_0, \frac{d}{dt} \frac{\partial u_0}{\partial p_1} \rangle \right) \\ &= \frac{1}{\sqrt{2}} \int_0^1 \langle -\frac{\ddot u_0}{T_0} - 2\nu' i \dot u_0 + T_0 \nabla \Phi_{\nu',\varepsilon}(u_0), \frac{\partial u_0}{\partial p_1} \rangle + \frac{1}{\sqrt{2}} \left[\langle \frac{\dot u_0(t)}{T_0} + \nu' i u_0(t), \frac{\partial u_0}{\partial p_1}(t) \rangle \right]_0^1 \\ &= \frac{1}{\sqrt{2}} \left[\langle \dot y_0(t) + \nu' y_0(t), \frac{\partial y_0}{\partial p_1}(t) \rangle \right]_0^{T(p_1)}. \end{aligned}$$

In the second equality we use the Jacobi constant for y_0, in the last one we use the fact that y_0 is a classical solution of the motion equation.
As in the step 3) of the proof of Theorem 5.3 of [11], we can compute

$$\frac{\partial}{\partial p_1} y_0(0) = 0 \qquad \frac{\partial}{\partial p_1} y_0(T(p_1)) = Id_{T_{p_1}(\partial B_R(0))}.$$

Hence

$$\frac{\partial}{\partial p_1} L_{\nu'}\left([0,T(p_1)];y_0\right)[\varphi] = \frac{1}{\sqrt{2}}\left(\langle \dot{y}_0(T(p_1)),\varphi\rangle + \nu'\langle ip_1,\varphi\rangle\right).$$

We can repeat the same computations for the second term in the right side of the (36), with minor changes: terming $\widetilde{y}_1 = \widetilde{y}(\cdot\,; p_1, \widetilde{p}_1; \varepsilon, \nu')$, we obtain

$$\frac{\partial}{\partial p_1} L_{\nu'}\left([0,T(p_1,\widetilde{p})_1];\widetilde{y}_1\right)[\varphi] = -\frac{1}{\sqrt{2}}\left(\langle \dot{\widetilde{y}}_1(0),\varphi\rangle + \nu'\langle ip_1,\varphi\rangle\right). \qquad \square$$

Lemma 5.6. *There exist $\bar{\varepsilon} > 0$ and $\bar{\nu}' > 0$ such that if $\varepsilon \in (0,\bar{\varepsilon})$ and $|\nu'| < \bar{\nu}'$ then*

$$\bar{p}_k \text{ minimizes } G_k \Rightarrow \bar{p}_k \in D_k^\circ \qquad \forall k.$$

The values $\bar{\varepsilon}$ and $\bar{\nu}'$ are independent on $(P_{k_1},\dots,P_{k_n}) \in \mathcal{P}^n$.

Proof. Adapt the proof of Lemma 3 in [12]. $\square$

Conclusion of the proof of Proposition 5.2. We can follow step 5) of the proof of Theorem 5.3 of [11] in order to check that each ζ_k is smooth. Recalling the construction of $\gamma^{((P_{k_1},\dots,P_{k_n}),\varepsilon,\nu')}$, the proof is complete. $\square$

6 Collision-free weak solutions

We will work with $\varepsilon \in (0,\bar{\varepsilon})$ which is fixed. The aim is to find a threshold $\bar{\nu}'_{th}(\varepsilon)$ such that, if $|\nu'| < \bar{\nu}'_{th}(\varepsilon)$, then $\gamma_{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ is collision-free. It is necessary to distinguish among:

$$1)\ \alpha = 1 \text{ and } N \geq 4, \qquad 2)\ \alpha = 1 \text{ and } N = 3 \qquad 3)\ \alpha \in (1,2).$$

1) $\alpha = 1$ and $N \geq 4$. We start by looking at Theorem 5.3 of [11]. Since $N \geq 4$, we have a simple way to choose $(P_{j_1},\dots,P_{j_n})$ so that the weak solution $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,0)}$ is a collision-free solution of the N-centre problem $\ddot{y} = \nabla V_\varepsilon(y)$, with energy -1: it is sufficient to take $P_{j_k} \in \mathcal{P}\setminus\mathcal{P}_1$ for every $k = 1,\dots,n$. Indeed in such a situation the conditions (ii)-$b)$ or (ii)-$c)$ of the quoted statement cannot be satisfied. Note that if $N = 3$ the set $\mathcal{P}\setminus\mathcal{P}_1$ is empty, and this is way that case deserves a different discussion. Now, let $\varepsilon \in (0,\bar{\varepsilon})$, $\nu' \in (-\bar{\nu}',\bar{\nu}')$, $n \in \mathbb{N}$ and $(P_{j_1},\dots,P_{j_n}) \in (\mathcal{P}\setminus\mathcal{P}_1)^n$; let $(\bar{p}_0,\dots,\bar{p}_{2n})$ be the minimizer of $F_{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ found in Proposition 5.2, and let $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ be the corresponding periodic weak solution of (8). Is it true that, for ν' sufficiently small, such a solution is still collision-free? The answer is affirmative: the idea is that if $\nu' \to 0$ the "minimizers" $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ are weakly convergent in H^1 to $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,0)}$, which is collision-free. This is true in a local sense, and can be considered as a kind of Gamma-convergence argument.

Continuity Lemma 6.1. *Let $\varepsilon \in (0,\bar{\varepsilon})$, $P_j \in \mathcal{P}$, $((p_1^m,p_2^m)) \subset (\partial B_R(0))^2$ and $(\nu'_m) \subset (-\bar{\nu}',\bar{\nu}')$. Let $u_m = u_{P_j}(\cdot\,; p_1^m, p_2^m; \varepsilon, \nu'_m)$ be a minimizer for the following variational problem:*

$$\min\left\{M_{\nu'_m}(u) : u \in K_{P_j}^{p_1^m p_2^m}([0,1])\right\}.$$

Assume $(p_1^m,p_2^m) \to (\widetilde{p}_1,\widetilde{p}_2)$, $\nu'_m \to 0$, and $u_m \rightharpoonup \widetilde{u}$ weakly in H^1. Then $\widetilde{u}$ is a minimizer for

$$\min\left\{M_0(u) : u \in K_{P_j}^{\widetilde{p}_1\widetilde{p}_2}([0,1])\right\}.$$

We postpone the proof of this Continuity Lemma in the next section; now, as announced, we use it in order to prove the following Proposition, which is the last step in the proof of Theorem 1.1 (recall Proposition 2.1 and Remark 2).

Proposition 6.2. *Let $\alpha = 1$ and $N \geq 4$. Let $\varepsilon \in (0, \bar{\varepsilon})$. There exists $\bar{\nu}_1'(\varepsilon)$ such that for every $\nu' \in (-\bar{\nu}_1'(\varepsilon), \bar{\nu}_1'(\varepsilon))$, $n \in \mathbb{N}$ and $(P_{j_1}, \dots, P_{j_n}) \in (\mathcal{P} \setminus \mathcal{P}_1)^n$, the function $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ is collision-free.*

Proof. Let $(P_{j_1}, \dots, P_{j_n}) \in (\mathcal{P} \setminus \mathcal{P}_1)^n$ and $\nu' \in (-\bar{\nu}', \bar{\nu}')$. The key observation is the following: when $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ stays inside $B_R(0)$, it coincides with a re-parameterization of an inner minimizer $u_{P_j}(\cdot\,; p_1, p_2; \varepsilon, \nu')$, for some p_1, p_2 and P_j. Therefore the thesis follows if we show that there exist $\bar{\nu}_1' = \bar{\nu}_1'(\varepsilon), \beta_1 = \beta_1(\varepsilon) > 0$ such that

$$\min_{k \in \{1,\dots,N\}} \left(\min_{t \in [0,1]} |u_{P_j}(t; p_1, p_2; \varepsilon, \nu') - c_k| \right) \geq \beta_1 \tag{37}$$

for every $(p_1, p_2, P_j, \nu') \in (\partial B_R(0))^2 \times (\mathcal{P} \setminus \mathcal{P}_1) \times (-\bar{\nu}_1', \bar{\nu}_1')$.
Assume by contradiction that this claim is not true. Then there are $(\beta_m) \subset \mathbb{R}^+$, $(\nu_m') \subset (-\bar{\nu}', \bar{\nu}')$, $((p_1^m, p_2^m)) \subset (\partial B_R(0))^2$, $(P_j^m) \subset (\mathcal{P} \setminus \mathcal{P}_1)$ and $(k_m) \subset \{1, \dots, N\}$ such that $\beta_m \to 0$, $\nu_m' \to 0$ for $m \to \infty$, and

$$\min_{t \in [0,1]} |u_{P_j^m}(t; p_1^m, p_2^m; \varepsilon, \nu_m') - c_{k_m}| = \beta_m \qquad \forall m.$$

Since $\{1, \dots, N\}$ and $\mathcal{P} \setminus \mathcal{P}_1$ are discrete and finite, we can assume $k_m = k$ and $P_j^m = P_j$ for every m. Also, since $\partial B_R(0)$ is compact, up to a subsequence $(p_1^m, p_2^m) \to (\widetilde{p}_1, \widetilde{p}_2) \in \partial B_R(0)$. We term $u_m = u_{P_j}(\cdot\,; p_1^m, p_2^m; \varepsilon, \nu_m')$. The set of the minimizers $\mathcal{IM}_\varepsilon$ is bounded in the H^1 norm, therefore up to a subsequence $u_m \rightharpoonup \widetilde{u} \in K_{P_j}^{\widetilde{p}_1 \widetilde{p}_2}([0,1])$ weakly in H^1 (and hence uniformly). In particular, the function $\widetilde{u}$ has at least one collision. The Continuity Lemma 6.1 implies that $\widetilde{u}$ is a collision minimizer of M_0 in $K_{P_j}^{\widetilde{p}_1 \widetilde{p}_2}([0,1])$; this is in contradiction with Theorem 4.12 of [11], since $P_j \notin \mathcal{P}_1$ (recall Remark 5). □

Remark 12. The Continuity Lemma permits to restrict the attention on a unique passage inside $B_R(0)$; in particular the argument is independent on n, which can be arbitrarily large.

2) $\alpha = 1$ and $N = 3$. This is the hardest part, since if we look at Theorem 5.3 of [11] we realize that it is not immediate to give conditions on $(P_{j_1}, \dots, P_{j_n})$ to obtain a collision-free periodic solution $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,0)}$ for the fixed energy N-centre problem

$$\begin{cases} \ddot{y}(t) = \nabla V_\varepsilon(y(t)) \\ \frac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)) = -1. \end{cases}$$

In order to work with a set of symbols such that the corresponding solutions are collision-free, we introduced $\mathcal{G}$ (see section 1); for every n and for every $(P_{j_1}, \dots, P_{j_{4n}}) \in \mathcal{G}^n$, the weak solution $\gamma^{((P_{j_1},\dots,P_{j_{4n}}),\varepsilon,0)}$ of the N-centre problem is actually a classical solution, because no composed sequence of elements of $\mathcal{G}$ has the reflection symmetry which characterizes a collision trajectory (see the following Remark 13). For $\varepsilon \in (0, \bar{\varepsilon})$, we aim at showing that, if $|\nu'|$ is sufficiently small, for every $n \in \mathbb{N}$ and $(P_{j_1}, \dots, P_{j_{4n}}) \in \mathcal{G}^n$ the function $\gamma^{((P_{j_1},\dots,P_{j_{4n}}),\varepsilon,\nu')}$ is still collision-free. The idea for the proof is exactly the same which we have already used in points 1) and 2). Unfortunately, while therein we can simply restrict our attention to the behaviour of any inner minimizer (that is a local

argument), here this approach does not work. Indeed, for every $P_j \in \mathcal{P}$ and $p_1 \in \partial B_R(0)$ it is possible that a minimizer of M_0 in $K_{P_j}^{p_1 p_1}([0,1])$ has collisions. Therefore we have to use an argument which is local, "but not too much".

Remark 13. A possible way to check that there aren't collisions for solutions to the 3-centre problem associated to sequences of partitions of $\mathcal{G}$ is the following. Let $\gamma^{((P_{k_1},\dots,P_{k_{4n}}),\varepsilon,0)}$ be the periodic solution of the N-centre problem found in Theorem 5.3 of [11]. Writing $(P_{k_1},\dots,P_{k_{4n}}) \in \mathcal{G}^n$ as an infinite periodic sequence, a group of 5 consecutive partitions is one of the following:

$$\begin{array}{llllll} P_1P_1P_2P_3P_1 & P_1P_1P_2P_3P_2 & P_1P_2P_3P_1P_1 & P_1P_2P_3P_2P_2 & P_2P_3P_1P_1P_2 & P_2P_3P_2P_2P_3 \\ & P_3P_1P_1P_2P_3 & P_3P_2P_2P_3P_1 & P_2P_2P_3P_1P_1 & P_2P_2P_3P_1P_2 & P_2P_3P_1P_1P_1 \\ & P_2P_3P_1P_2P_2 & P_3P_1P_1P_1P_2 & P_3P_1P_2P_2P_3 & P_1P_1P_1P_2P_3 & P_1P_2P_2P_3P_1. \end{array} \tag{38}$$

For instance, assume that $(P_{k_1},\dots,P_{k_5}) = (P_2,P_3,P_1,P_1,P_2)$ (this is possible even if no element of $\mathcal{G}$ begins with P_2,P_3, because we have to consider the possibility of applying the right shift a finite number of time; in this case, since $(P_{k_1},\dots,P_{k_{4n}})$ has been built by the juxtaposition of elements of $\mathcal{G}$, the last two symbols $P_{k_{4n-1}},P_{k_{4n}}$ have to be P_1,P_1); let us check that $\gamma^{((P_{k_1},\dots,P_{k_{4n}}),\varepsilon,0)}$ cannot have a collision in its third passage inside $B_R(0)$. By construction there are $\bar{p}_1,\bar{p}_{10} \in \partial B_R(0)$ such that $\gamma = \gamma^{((P_{k_1},\dots,P_{k_{4n}}),\varepsilon,0)}$ (up to a time translation) starts from $\bar{p}_1$ with velocity directed inwards $B_R(0)$, crosses 5 times $B_R(0)$ separating the centres according to the five partitions prescribed in succession and arrives in $\bar{p}_{10}$. If γ has a collision during its third passage inside $B_R(0)$, the inner trajectory describing this third passage has to be an ejection-collision solution, so that the second and the fourth symbols have to be equal for a sake of symmetry, but this is not true: $P_3 \neq P_1$. A collision cannot occur even in the second passage: there are $\bar{p}_{-5},\bar{p}_6 \in \partial B_R(0)$ such that γ starts from $\bar{p}_{-5}$ with velocity directed inwards $B_R(0)$, crosses 5 times $B_R(0)$ separating the centres according to the five partitions $(P_{k_{4n}},P_{k_1},\dots,P_{k_4}) = (P_1,P_2,P_3,P_1,P_1)$ in succession and arrives in $\bar{p}_6$; if γ collides in c_3 in the third passage, the inner trajectory describing the third passage inside $B_R(0)$ has to be an ejection-collision solution, so that the second and the fourth symbols have to be equal, which is not true. We can iterate this line of reasoning.

We collect the possible groups of 5 consecutive partitions in (38) in a set $\widetilde{\mathcal{P}}^5 \subset \mathcal{P}$. Let us fix $\varepsilon \in (0,\bar{\varepsilon})$, $p_1,p_{10} \in \partial B_R(0)$, $(P_{k_1},\dots,P_{k_5}) \in \widetilde{\mathcal{P}}^5$, $\nu' \in (-\bar{\nu},\bar{\nu})$. Let

$$B := \{(p_2,\dots,p_9) \in (\partial B_R(0))^8 : |p_{2j}-p_{2j+1}| \le \delta,\ j=1,\dots,4\}.$$

As we associated to each point of D a periodic function, to each point of B we can associate a (non-periodic) function in the following way. For each $j=1,\dots,4$ we can connect p_{2j} and p_{2j+1} with an outer solution $y_{2j} = y_{\mathrm{ext}}(\cdot\,;p_{2j},p_{2j+1};\varepsilon,\nu')$ of (15); for each $j=0,\dots,4$ we can connect p_{2j+1} and p_{2j+2} with an inner solution $y_{2j+1} = y_{P_{k_{j+1}}}(\cdot\,;p_{2j+1},p_{2j+2};\varepsilon,\nu')$ of (26). We set $t_1 := 0$, $t_k := \sum_{j=1}^{k-1} T_j$ for $k=2,\dots,10$, where $[0,T_j]$ is the time interval of y_j. We define

$$\sigma_{(p_2,\dots,p_9)}^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}(t) := \begin{cases} y_1(t) & t \in [t_1,t_2] \\ y_2(t-t_2) & t \in [t_2,t_3] \\ \vdots & \\ y_9(t-t_9) & t \in [t_9,t_{10}]. \end{cases} \tag{39}$$

By the definition $\sigma_{(p_2,\dots,p_9)}^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}(t_k) = p_k$. We introduce a function $\mathfrak{F}_{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')} : B \to \mathbb{R}$ as

$$\mathfrak{F}_{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}(p_2,\dots,p_9) := L_{\nu'}\left([0,t_{10}];\sigma_{(p_2,\dots,p_9)}^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}\right).$$

Note the analogy between the definition of $\mathfrak{F} = \mathfrak{F}_{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}$ and $F = F_{((P_{k_1},\dots,P_{k_n}),\varepsilon,\nu')}$. The function $\mathfrak{F}$ is continuous on the compact set B (apply the same proof already used for the continuity of F), therefore it has a minimum. We denote by $\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}$ the glued function associated to an arbitrarily chosen minimizer.
Let $(P_{k_1},\dots,P_{k_{4n}}) \in \mathcal{G}^n$. The following Lemma relates the minimality properties of F and of $\mathfrak{F}$; in what follows the indexes have to be considered by periodicity: for instance writing $2j+5$ we mean $2j+5 \mod 8n$.

Lemma 6.3. *Let $(\bar{p}_0,\dots,\bar{p}_{8n}) \in D$ be a minimizer of $F_{((P_{k_1},\dots,P_{k_{4n}}),\varepsilon,\nu')}$. Then, for every $j = 0,\dots,4n-1$, the point $(\bar{p}_{2j+2},\dots,\bar{p}_{2j+9}) \in B$ is a minimizer of $\mathfrak{F}_{((\bar{p}_{2j+1},\bar{p}_{2j+10}),(P_{k_{j+1}},\dots,P_{k_{j+5}}),\varepsilon,\nu')}$. In particular*

$$\gamma_{((P_{k_{j+1}},\dots,P_{k_{j+5}}),\varepsilon,\nu')}|_{[\mathfrak{T}_{2j},\mathfrak{T}_{2j+10}]} \equiv \sigma^{((\bar{p}_{2j+1},\bar{p}_{2j+10}),(P_{k_{j+1}},\dots,P_{k_{j+5}}),\varepsilon,\nu')}.$$

Proof. It is an immediate consequence of the additivity of the functional $L_{\nu'}$. □

As a consequence, the following statement can be proved applying the same argument already explained in Remark 13.

Lemma 6.4. *Let $\varepsilon \in (0,\bar{\varepsilon})$. For every $((p_1,p_{10}),(P_{k_1},\dots,P_{k_5})) \in (\partial B_R(0))^2 \times \widetilde{\mathcal{P}}^5$ the function $\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,0)}$ is collision-free during its third passage inside the ball $B_R(0)$.*

We denote with $T(\sigma)$ or $T^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)}$ the maximum of the time interval of $\sigma = \sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)}$. We collect the boundedness properties of outer and inner solutions, see Lemma 3.4 and Corollary 4.11.

Lemma 6.5. *Let $\varepsilon \in (0,\bar{\varepsilon})$. There are $C_1, C_2, C_3 > 0$ such that*

$$C_1 \le T^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)} \le C_2$$
$$\|\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)}\|_{H^1([0,T(\sigma)])} \le C_3$$

for every $((p_2,\dots,p_9),(p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\nu') \in B \times (\partial B_R(0))^2 \times \widetilde{\mathcal{P}}^5 \times (-\bar{\nu}',\bar{\nu}')$.

It is preferable to deal with functions defined in the same time interval. Therefore, for every $\sigma = \sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)}$ we introduce the re-parameterization $v(t) := v^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)}(t) = \sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)}(T(\sigma)t)$, for $t \in [0,1]$.

Definition 5. We collect the "glued function" v in

$$\mathcal{GF}_\varepsilon := \Big\{ v = v^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)} \text{for some } (p_2,\dots,p_9) \in B,$$
$$(p_1,p_{10}) \in (\partial B_R(0))^2,\ (P_{k_1},\dots,P_{k_5}) \in \widetilde{\mathcal{P}}^5,\ |\nu'| < \bar{\nu}' \Big\}.$$

For each $v \in \mathcal{GF}_\varepsilon$ we term

$$\omega(v)^2 := \frac{\int_0^1 \Phi_{\nu',\varepsilon}(v) - 1}{\frac{1}{2}\int_0^1 |\dot{v}|^2}.$$

Note that, if $v(t) = \sigma(T(\sigma)t)$, then $\omega(v) = 1/T(\sigma)$. Note also that for every $\varepsilon \in (0, \bar{\varepsilon})$ there exists $C > 0$ such that $\|v\|_{H^1} \leq C$ for every $v \in \mathcal{GF}_\varepsilon$. It follows from Lemma 6.5, taking into account the boundedness properties for the time intervals of inner and outer solutions. In order to work with sequences of functions in $\mathcal{GF}_\varepsilon$, it is convenient to introduce some notation. Fixed $(P_{k_1}, \ldots, P_{k_5}) \in \mathcal{P}^5$ and $\varepsilon \in (0, \bar{\varepsilon})$, assume that we have $((p_2^m, \ldots, p_9^m))_m \subset B$, $((p_1^m, p_{10}^m))_m \subset (\partial B_R(0))^2$, $(\nu'_m) \subset (-\bar{\nu}', \bar{\nu}')$ such that

$$(p_2^m, \ldots, p_9^m) \to (\widehat{p}_2, \ldots, \widehat{p}_9) \qquad (p_1^m, p_{10}^m) \to (\widehat{p}_1, \widehat{p}_{10}) \qquad \nu'_m \to 0.$$

We will use the following notations

$$v_m := v_{(p_2^m, \ldots, p_9^m)}^{((p_1^m, p_{10}^m), (P_{k_1}, \ldots, P_{k_5}), \varepsilon, \nu'_m)} \qquad \omega_m := \omega(v_m) \tag{40}$$

$$\sigma_m := \sigma_{(p_2^m, \ldots, p_9^m)}^{((p_1^m, p_{10}^m), (P_{k_1}, \ldots, P_{k_5}), \varepsilon, \nu'_m)} \qquad T_m := T(\sigma_m); \tag{41}$$

Subscripts will be replaced by the accent $\widehat{\cdot}$ for the function corresponding to the limit points. Recall that σ_m has been obtained by the juxtaposition of

$$y_{P_{k_{j+1}}}(\cdot\,; p_{2j+1}^m, p_{2j+2}^m; \varepsilon, \nu'_m) =: y_{2j+1}^m \quad \text{and} \quad y_{\text{ext}}(\cdot\,; p_{2j}^m, p_{2j+1}^m; \varepsilon, \nu'_m) =: y_{2j}^m.$$

Each y_j^m is defined over a time interval $[0, T_j^m]$. There are $0 = t_1^m < t_2^m < \ldots t_9^m < t_{10}^m = T(\sigma_m)$ such that $\sigma_m(t_k^m) = p_k^m$ for every $k = 1, \ldots, 10$. We have $T_j^m = t_{j+1}^m - t_j^m$. For $j = 0, \ldots, 4$, recall that

$$y_{P_{k_{j+1}}}(\cdot\,; p_{2j+1}^m, p_{2j+2}^m; \varepsilon, \nu'_m) = u_{P_{k_{j+1}}}\left(\frac{\cdot}{T_{2j+1}^m}\cdot\,; p_{2j+1}^m, p_{2j+2}^m; \varepsilon, \nu'_m\right) =: u_{2j+1}^m.$$

Lemma 6.6. *Let $\varepsilon \in (0, \bar{\varepsilon})$, $(P_{k_1}, \ldots, P_{k_5}) \in \mathcal{P}^5$. Assume that we have sequences $((p_2^m, \ldots, p_9^m))_m \subset B$, $((p_1^m, p_{10}^m))_m \subset (\partial B_R(0))^2$, $(\nu'_m) \subset (-\bar{\nu}', \bar{\nu}')$ such that*

$$(p_2^m, \ldots, p_9^m) \to (\widehat{p}_2, \ldots, \widehat{p}_9) \qquad (p_1^m, p_{10}^m) \to (\widehat{p}_1, \widehat{p}_{10}) \qquad \nu'_m \to 0.$$

Using the notations previously introduced, assume that exists $v \in H^1([0,1])$ such that $v_m \rightharpoonup v$ weakly in H^1. Then

$$v = v_{(\widehat{p}_2, \ldots, \widehat{p}_9)}^{((\widehat{p}_1, \widehat{p}_{10}), (P_{k_1}, \ldots, P_{k_5}), \varepsilon, 0)}.$$

Proof. Under the convergence of the ends and of ν'_m, inner and outer solutions y_k^m are weakly convergent to inner and outer solutions $\widehat{y}_k$ (see Propositions 3.1 and the Continuity Lemma 6.1); the thesis follows easily. $\square$

To each $((p_1, p_{10}), (P_{k_1}, \ldots, P_{k_5}), \nu') \in (\partial B_R(0))^2 \times \widetilde{\mathcal{P}}^5 \times (-\bar{\nu}'_1, \bar{\nu}'_1)$ we can associate an element of $\mathcal{GF}_\varepsilon$ in the following way: it is well defined the function $\mathfrak{F}_{((p_1, p_{10}), (P_{k_1}, \ldots, P_{k_5}), \varepsilon, \nu')}$, and we know that it has a minimum. To a minimum we associated the function $\sigma^{((p_1, p_{10}), (P_{k_1}, \ldots, P_{k_5}), \varepsilon, \nu')}$, which can be re-parametrized obtaining $v^{((p_1, p_{10}), (P_{k_1}, \ldots, P_{k_5}), \varepsilon, \nu')}$. We are ready to state the counterpart of the Continuity Lemma 6.1.

Continuity Lemma 6.7. *Let $\varepsilon \in (0, \bar{\varepsilon})$, $(P_{k_1}, \ldots, P_{k_5}) \in \mathcal{P}^5$, $((p_1^m, p_{10}^m)) \subset (\partial B_R(0))^2$ and $(\nu'_m) \subset (-\bar{\nu}', \bar{\nu}')$. Let $v_m = v^{((p_1^m, p_{10}^m), (P_{k_1}, \ldots, P_{k_5}), \varepsilon, \nu'_m)}$ be a function of $\mathcal{GF}_\varepsilon$ associated to a minimizer of the following variational problem:*

$$\min\left\{\mathfrak{F}_{((p_1^m, p_{10}^m), (P_{k_1}, \ldots, P_{k_5}), \varepsilon, \nu'_m)}(p_2, \ldots, p_9) : (p_2, \ldots, p_9) \in B\right\}.$$

Assume $(p_1^m, p_{10}^m) \to (\widetilde{p}_1, \widetilde{p}_{10})$, $\nu'_m \to 0$, and $v_m \rightharpoonup \widetilde{v}$ weakly in H^1. Then $\widetilde{v}$ is the function associated to a minimizer for

$$\min\left\{\mathfrak{F}_{((\widetilde{p}_1,\widetilde{p}_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,0)}(p_2,\dots,p_9) : (p_2,\dots,p_9)\in B\right\}.$$

This continuity result permits to prove the following Proposition, which is the last step in the proof of Theorem 1.2.

Proposition 6.8. *Let $\alpha = 1$ and $N = 3$. Let $\varepsilon \in (0,\bar{\varepsilon})$. There exists $\bar{\nu}'_2(\varepsilon)$ such that for every $\nu' \in (-\bar{\nu}'_2(\varepsilon), \bar{\nu}'_2(\varepsilon))$, $n \in \mathbb{N}$ and $(P_{j_1},\dots,P_{j_{4n}}) \in \mathcal{G}^n$, the function $\gamma^{((P_{j_1},\dots,P_{j_{4n}}),\varepsilon,\nu')}$ is collision-free.*

Proof. Let $(P_{j_1},\dots,P_{j_{4n}}) \in \mathcal{G}^n$ and $\nu' \in (-\bar{\nu}', \bar{\nu}')$. Let us consider the restriction of $\gamma = \gamma^{((P_{j_1},\dots,P_{j_{4n}}),\varepsilon,\nu')}$ in a time interval $[s_1,s_2]$, chosen in such a way that $\gamma|_{[s_1,s_2]}$ describes one passage of γ inside $B_R(0)$. The goal is to show that $\gamma|_{[s_1,s_2]}$ is collision-free. There are

- $t_k \in \mathbb{R}$ and $p_k \in \partial B_R(0)$ such that $\gamma(t_k) = p_k$, for every $k = 1,\dots,10$.
- $(P_{k_1},\dots,P_{k_5}) \in \mathcal{P}^5$,

such that $\gamma|_{[t_1,t_{10}]} = \sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')} = \sigma$ and $\gamma|_{[s_1,s_2]} = \sigma|_{[t_5,t_6]}$, where t_5 and t_6 have been defined in (39). This means that each passage of γ inside $\partial B_R(0)$ is the third passage inside $\partial B_R(0)$ of a function $\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}$, for some $p_1, p_{10} \in \partial B_R(0)$ and $(P_{k_1},\dots,P_{k_5}) \in \mathcal{P}^5$. This observation is the key point of the proof: it implies that our thesis follows if we show that there are $\bar{\nu}'_2, \beta_2 > 0$ such that

$$\min_{k\in\{1,\dots,N\}}\left(\min_{t\in\left[\frac{t_5}{T(\sigma)},\frac{t_6}{T(\sigma)}\right]} |v^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}(t) - c_{k_3}|\right) \geq \beta_2 \tag{42}$$

for every $((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\nu') \in (\partial B_R(0))^2 \times \widetilde{\mathcal{P}}^5 \times (-\bar{\nu}'_2, \bar{\nu}'_2)$; this implies that $v^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}$ (and hence $\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}$) cannot have a collision in its third passage inside $B_R(0)$, independently on (p_1,p_{10}) and $(P_{k_1},\dots,P_{k_5})$.
Assume by contradiction that (42) is not true. Then there are $(\beta_m) \subset \mathbb{R}^+$, $(\nu'_m) \subset (-\bar{\nu}', \bar{\nu}')$, $((p_1^m, p_2^m)) \subset (\partial B_R(0))^2$, $((P_{k_1},\dots,P_{k_5})^m) \subset \widetilde{\mathcal{P}}^5$ such that $\beta_m \to 0$, $\nu'_m \to 0$ for $m \to \infty$, and

$$\min_{t\in\left[\frac{t_5^m}{T(\sigma_m)},\frac{t_6^m}{T(\sigma_m)}\right]} |v^{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5})^m,\varepsilon,\nu'_m)}(t) - c_{k_3^m}| = \beta_m \qquad \forall m.$$

Since $\widetilde{\mathcal{P}}^5$ is discrete and finite, we can assume $(P_{k_1},\dots,P_{k_5})^m = (P_{k_1},\dots,P_{k_5})$ for every m. Also, since $\partial B_R(0)$ is compact, up to a subsequence $(p_1^m, p_2^m) \to (\widehat{p}_1, \widehat{p}_2) \in \partial B_R(0)$. We term $v_m = v^{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5})^m,\varepsilon,\nu'_m)}$. The image of v_m intersects the circle $\partial B_R(0)$ in 8 points $(p_2^m,\dots,p_9^m) \in B$ in succession. Up to a subsequence $(p_2^m,\dots,p_9^m) \to (\widehat{p}_2,\dots,\widehat{p}_9)$. We observed that the set $\mathcal{GF}_\varepsilon$ is bounded in the H^1 norm, therefore up to a subsequence $v_m \rightharpoonup \widehat{v} \in H^1([0,1])$ weakly in H^1 (and hence uniformly). The image of $\widehat{v}$ intersects the circle in the 8 points $(\widehat{p}_2,\dots,\widehat{p}_9)$ in succession. To be precise

$$\widehat{v} = v_{(\widehat{p}_2,\dots,\widehat{p}_9)}^{((\widehat{p}_1,\widehat{p}_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,0)} \in \mathcal{GF}_\varepsilon,$$

see Lemma 6.6. By the Continuity Lemma 6.7, the point $(\widehat{p}_2,\dots,\widehat{p}_9)$ minimizes $\mathfrak{F}_{((\widehat{p}_1,\widehat{p}_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,0)}$ in B. But the uniform convergence implies that $\widehat{v}$ has a collision in its third passage inside $B_R(0)$, and this is in contradiction with Lemma 6.4. $\square$

3) $\alpha \in (1,2)$**.** This is the easiest case, since for every $\varepsilon \in (0,\bar{\varepsilon})$, $n \in \mathbb{N}$, $(P_{j_1},\dots,P_{j_n}) \in \mathcal{P}^n$ the weak solution $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,0)}$ is collision-free (Theorem 5.3 of [11]). Thus, we can simply follows the sketch already developed for point 1) with minor changes.

Proposition 6.9. *Let $\alpha \in (1,2)$. Let $\varepsilon \in (0,\bar{\varepsilon})$. There exists $\bar{\nu}'_3(\varepsilon)$ such that for every $\nu' \in (-\bar{\nu}'_3(\varepsilon),\bar{\nu}'_3(\varepsilon))$, $n \in \mathbb{N}$ and $(P_{j_1},\dots,P_{j_n}) \in \mathcal{P}^n$, the function $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ is collision-free.*

7 Proofs of the continuity lemmas

7.1 Proof of Continuity Lemma 6.1

Let u_0 be a minimizer of M_0 in $K^{\widetilde{p}_1\widetilde{p}_2}_{P_j}([0,1])$. We aim at proving that $M_0(\widetilde{u}) = M_0(u_0)$. We will briefly write L_m for $L_{\nu'_m}$ and M_m for $M_{\nu'_m}$.
The following statement is a continuity property for the functionals $\{M_m\}$ in the set of the minimizers $\{u_m\}$.

Lemma 7.1. *The family $\{M_m\}_m$ tends to M_0 as $m \to \infty$, uniformly in the set $\{u_m : m \in \mathbb{N}\}$. This means that for every $\lambda > 0$ exists $m_1 \in \mathbb{N}$ such that*

$$m > m_1 \Rightarrow |M_m(u_{\bar{m}}) - M_0(u_{\bar{m}})| \le \lambda \quad \forall \bar{m} \in \mathbb{N}.$$

Proof. Let $\bar{m} \in \mathbb{N}$. For every m we have

$$\begin{aligned}|M_m(u_{\bar{m}}) - M_0(u_{\bar{m}})| \le |\nu'_m| \int_0^1 |u_{\bar{m}}||\dot{u}_{\bar{m}}| \\ + \sqrt{2}\left(\int_0^1 |\dot{u}_{\bar{m}}|\right)^{\frac{1}{2}} \left| \left(\int_0^1 V_\varepsilon(u_{\bar{m}}) - 1 + \frac{(\nu'_m)^2}{2}|u_{\bar{m}}|^2\right)^{\frac{1}{2}} - \left(\int_0^1 V_\varepsilon(u_{\bar{m}}) - 1\right)^{\frac{1}{2}} \right|\end{aligned}$$

Let $\varphi_{\bar{m}}(\nu) := \left(\int_0^1 V_\varepsilon(u_{\bar{m}}) - 1 + \frac{(\nu^2)}{2}|u_{\bar{m}}|^2\right)^{1/2}$. It results

$$|\varphi_{\bar{m}}(\nu'_m) - \varphi_{\bar{m}}(0)| \le \frac{1}{2}\left(\int_0^1 V_\varepsilon(u_{\bar{m}}) - 1\right)^{-\frac{1}{2}} \int_0^1 |u_{\bar{m}}|^2(\nu'_m)^2 \le \frac{R^2}{2\sqrt{M_1}}(\nu'_m)^2,$$

so that

$$|M_m(u_{\bar{m}}) - M_0(u_{\bar{m}})| \le R\|\dot{u}_{\bar{m}}\|_{L^2}|\nu'_m| + \frac{R^2}{\sqrt{2M_1}}\|\dot{u}_{\bar{m}}\|_{L^2}(\nu'_m)^2 \le C(|\nu'_m| + (\nu'_m)^2),$$

where C is a constant independent on $\bar{m}$ (see Proposition 4.10). $\square$

We want to compare $M_m(u_m)$ with $M_m(u_0)$. Because of the minimality property of u_m it seems reasonable to think that $M_m(u_m) \le M_m(u_0)$. This is not immediate, and not necessarily true, since u_m is a minimizer of M_m for the fixed ends problem $\min\{M_m(u) : u \in K^{p_1^m p_2^m}_{P_j}([0,1])\}$, while u_0 connects $\widetilde{p}_1$ and $\widetilde{p}_2$. However, the fact that $p_1^m \to \widetilde{p}_1$ and $p_2^m \to \widetilde{p}_2$ suggests that maybe we can prove something similar (which in fact will be equation (45)). For every $p_*, p_{**} \in \partial B_R(0)$ we consider again the function $\zeta_R(\cdot\,; p_*, p_{**})$ which parametrizes the shorter arc of $\partial B_R(0)$ connecting p_* and p_{**} in time 1 with constant angular velocity. It is easy to check that

$$\forall \lambda > 0\ \exists \rho > 0 :\ |p_* - p_{**}| < \rho \Rightarrow M_0(\zeta_R(\cdot\,; p_*, p_{**})) < \lambda,$$

so that

$$\forall\lambda > 0\ \exists m_2 \in \mathbb{N}:\ m > m_2 \Rightarrow \begin{cases} M_0(\zeta_R(t; p_1^m, \widetilde{p}_1)) < \lambda \\ M_0(\zeta_R(t; \widetilde{p}_2, p_2^m)) < \lambda. \end{cases} \tag{43}$$

Furthermore, the following continuity property holds true.

Lemma 7.2. *The family $\{M_m\}_m$ tends to M_0 as $m \to \infty$, uniformly in the set $\{\zeta_R(\cdot\,; p_*, p_{**}) : p_*, p_{**} \in \partial B_R(0)\}$. This means that for every $\lambda > 0$ exists $m_3 \in \mathbb{N}$ such that*

$$m > m_3 \Rightarrow |M_m(\zeta_R(\cdot\,; p_*, p_{**})) - M_0(\zeta_R(\cdot\,; p_*, p_{**}))| \le \lambda \quad \forall p_*, p_{**} \in \partial B_R(0).$$

Proof. We can adapt the proof of Lemma 7.1 with minor changes. $\square$

Conclusion of the proof of the Continuity Lemma 6.1. Because of the minimality of u_0 and the weak lower semi-continuity of M_0 it results

$$M_0(u_0) \le M_0(\widetilde{u}) \le \liminf_{m\to\infty} M_0(u_m). \tag{44}$$

For every $m \in \mathbb{N} \cup \{0\}$ we have

$$\frac{\omega_m^2}{2}|\dot{u}_m|^2 - \Phi_{\nu'_m,\varepsilon}(u_m) = -1 \quad \text{a.e. in } [0,1] \ \Rightarrow\ \sqrt{2}L_m(u_m) = M_m(u_m),$$

where $\omega_m = \omega_{P_j}(p_1^m, p_2^m; \varepsilon, \nu'_m)$. The variational characterization of u_m implies that

$$\begin{aligned} M_m(u_m) = \sqrt{2}L_m(u_m) &\le \sqrt{2}L_m(\zeta_R(\cdot\,; p_1^m, \widetilde{p}_1)) + \sqrt{2}L_m(u_0) + \sqrt{2}L_m(\zeta_R(\cdot\,; \widetilde{p}_2, p_2^m)) \\ &\le M_m(\zeta_R(\cdot\,; p_1^m, \widetilde{p}_1)) + M_m(u_0) + M_m(\zeta_R(\cdot\,; \widetilde{p}_2, p_2^m)). \end{aligned} \tag{45}$$

We passed to the functional L_m in order to exploit its additivity property, which does not hold for M_m. Lemmas 7.1, 7.2 and equation (43) imply that for every $\lambda > 0$ if $m > \max\{m_1, m_2, m_3\}$ then

$$\begin{cases} M_m(u_m) > M_0(u_m) - \lambda \\ M_m(\zeta_R(\cdot\,; p_1^m, \widetilde{p}_1)) < M_0(\zeta_R(\cdot\,; p_1^m, \widetilde{p}_1)) + \lambda < 2\lambda \\ M_m(\zeta_R(\cdot\,; \widetilde{p}_2, p_2^m)) < M_0(\zeta_R(\cdot\,; \widetilde{p}_2, p_2^m)) + \lambda < 2\lambda \\ M_m(u_0) < M_0(u_0) + \lambda. \end{cases}$$

Hence, from equation (45), for every $\lambda > 0$ if $m > \max\{m_1, m_2, m_3\}$ then

$$M_0(u_m) - \lambda \le M_0(u_0) + 5\lambda \Rightarrow \limsup_{m\to\infty} M_0(u_m) \le M_0(u_0).$$

This, together with (44), says that the sequence $(M_0(u_m))_m$ has a limit and $M_0(u_0) = M_0(\widetilde{u}) = \lim_m M_0(u_m)$; in particular $\widetilde{u}$ is a minimizer of M_0 in $K_{P_j}^{\widetilde{p}_1\widetilde{p}_2}([0,1])$. $\square$

7.2 Proof of the Continuity Lemma 6.7

Let $\sigma_0 = \sigma^{((\widetilde{p}_1,\widetilde{p}_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,0)} = \sigma_{(\widehat{p}_2,\ldots,\widehat{p}_9)}^{((\widetilde{p}_1,\widetilde{p}_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,0)}$, where $(\widehat{p}_2,\ldots,\widehat{p}_9)$ is a minimizer of $\mathfrak{F}_{((\widetilde{p}_1,\widetilde{p}_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,0)}$, and let $v_0(t) = \sigma_0(T(\sigma_0 t))$. We aim at proving that $M_0(\widetilde{v}) = M_0(v_0)$. We need two intermediate results. The first one is a generalization of Lemma 7.1 for the glued functions.

Lemma 7.3. *Let $(v_m) \subset \mathcal{GF}_\varepsilon$, where each v_m is a glued function defined by (40). The family $\{M_m\}_m$ tends to M_0 for $m \to \infty$, uniformly in $\{v_m\}_m$. This means that for every $\lambda > 0$ exists $m_1 \in \mathbb{N}$ such that*

$$m > m_1 \Rightarrow |M_m(v_{\bar{m}}) - M_0(v_{\bar{m}})| < \lambda \qquad \forall \bar{m}.$$

Proof. We can adapt the proof of Lemma 7.3; the only difference is that we used the uniform bounds

$$\|u\|_{L^2} \leq R \qquad \|\dot{u}\|_{L^2} \leq C \qquad \int_0^1 V_\varepsilon(u) - 1 \geq M_1 \qquad \forall u \in \mathcal{IM}_\varepsilon.$$

Now we are considering glued functions, so we need similar properties for the function of $\mathcal{GF}_\varepsilon$. We have already noticed that there is $C > 0$ such that $\|\dot{v}_{\bar{m}}\|_{H^1} \leq C$ for every $\bar{m}$; furthermore,

$$\int_0^1 V_\varepsilon(v_{\bar{m}}) - 1 \geq \frac{1}{T(\sigma_{\bar{m}})} \sum_{j=1}^{4} \int_0^{T_{2j+1}} (V_\varepsilon(y_{2j+1}) - 1) \geq \frac{4M_1}{C}. \qquad \square$$

Lemma 7.4. *Let $p_{2j}, p_{2j+1} \in \partial B_R(0)$ be such that $|p_{2j} - p_{2j+1}| \leq \delta$, let $(\nu'_m) \subset (-\bar{\nu}', \bar{\nu}')$ be such that $\nu'_m \to 0$ as $m \to \infty$.*
For every $\lambda > 0$ there exists $m_4 = m_4(p_{2j}, p_{2j+1}) \in \mathbb{N}$ such that

$$|L_{\nu'_{\bar{m}}}(y_{ext}(\cdot\,; p_{2j}, p_{2j+1}; \varepsilon, \nu'_m)) - L_{\nu'_{\bar{m}}}(y_{ext}(\cdot\,; p_{2j}, p_{2j+1}; \varepsilon, 0))| < \lambda$$

for every $\bar{m} \in \mathbb{N}$.

Proof. We will write y_m instead of $y_{\text{ext}}(\cdot\,; p_{2j}, p_{2j+1}; \varepsilon, \nu'_m)$ to ease the notation. Let T_m be such that $y_m(T_m) = p_{2j+1}$.

$$\begin{aligned} |L_{\bar{m}}(y_m) - L_{\bar{m}}(y_0)| \leq & \left| \int_0^{T_m} \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(t)) - 1}|\dot{y}_m(t)|\,dt - \int_0^{T_0} \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(t)) - 1}|\dot{y}_0(t)|\,dt \right| \\ & + \left| \int_0^{T_m} \langle iy_m(t), \dot{y}_m(t)\rangle\,dt - \int_0^{T_0} \langle iy_0(t), \dot{y}_0(t)\rangle\,dt \right|. \end{aligned} \tag{46}$$

We have already observed (Remark 7) that $\int_0^1 \langle iu, \dot{u}\rangle$ is continuous in the weak topology of H^1. We know that $y_m \to y_0$ $\mathcal{C}^1$-uniformly; it is not difficult to check that consequently

$$y_m(T_m t) \to y_0(T_0 t) \quad \mathcal{C}^1\text{-uniformly in } [0,1], \tag{47}$$

so that the second term in the right hand side of (46) tends to 0 as $m \to \infty$ (independently on $\bar{m}$). As far as the first term on the right hand side of (46) is concerned, it results

$$\begin{aligned} &\left| \int_0^{T_m} \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(t)) - 1}|\dot{y}_m(t)|\,dt - \int_0^{T_0} \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(t)) - 1}|\dot{y}_0(t)|\,dt \right| \\ &\qquad \leq \int_0^1 \left| \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - 1} - \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1} \right| |\dot{y}_m(T_m t)|\,dt \\ &\qquad\qquad + \int_0^1 \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1}\, \big||\dot{y}_0(T_0 t)| - |\dot{y}_m(T_m t)|\big|\,dt. \end{aligned} \tag{48}$$

The function $\sqrt{\cdot}$ is 1/2-Hölder continuous, so that for every $\bar{m}$

$$\begin{aligned}
&\int_0^1 \left|\sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - 1} - \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1}\right| |\dot{y}_m(T_m t)|\, dt \\
&\qquad \leq \left(\int_0^1 \left|\sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - 1} - \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1}\right|^2 dt\right)^{\frac{1}{2}} \|\dot{y}_m(T_m \cdot)\|_{L^2} \\
&\qquad \leq C \left(\int_0^1 |\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - \Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t))|\, dt\right)^{\frac{1}{2}}; \qquad (49)
\end{aligned}$$

In the last inequality, we took advantage of the uniform bound for the L^2 norm of outer solutions. Both y_m and y_0 are outer solutions, therefore we can exploit the fact that V_ε is $\mathcal{C}^\infty$ with bounded derivatives outside $\partial B_R(0)$; using also (47) and the first estimate (16), we obtain

$$\sup_{t\in[0,1]} |\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - \Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t))| \leq C(1 + |\nu'_{\bar{m}}|^2) \sup_{t\in[0,1]} |y_m(T_m t) - y_0(T_0 t)| \to 0 \tag{50}$$

as $m \to \infty$, independently on $\bar{m}$ (recall that $|\nu'_{\bar{m}}| \leq \bar{\nu}'$). Furthermore, using again (47) it is easy to check

$$\int_0^1 \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1}\, ||\dot{y}_0(T_0 t)| - |\dot{y}_m(T_m t)||\, dt \to 0, \tag{51}$$

as $m \to \infty$, independently on $\bar{m}$. Collecting (49), (50), (51) and comparing with (48) we deduce that also the first term on the right hand side of (46) tends to 0, uniformly in $\bar{m}$. □

Conclusion of the proof of the Continuity Lemma 6.7. The conservation of the Jacobi constant holds true both for v_0 and $\widetilde{v}$ (recall that $\widetilde{v} \in \mathcal{GF}_\varepsilon$, as showed in Lemma 6.6); using this property, the minimality of σ_0 and the weak lower semi-continuity of M_0, we have

$$M_0(v_0) = L_0(v_0) \leq L_0(\widetilde{v}) = M_0(\widetilde{v}) \leq \liminf_{m\to\infty} M_0(v_m). \tag{52}$$

We pose $\widehat{p}_1 := \widetilde{p}_1$ and $\widehat{p}_{10} := \widetilde{p}_{10}$. The minimality of $(p_2^m, \dots, p_9^m)$ for $\mathfrak{F}_{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu'_m)}$ implies that

$$\begin{aligned}
M_m(\sigma_m) &= \sqrt{2}L_m(\sigma_m) \leq \sqrt{2}L_m(\sigma^{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu'_m)}_{(\widehat{p}_2,\dots,\widehat{p}_9)}) \\
&\leq \sqrt{2}\left(L_m(\sigma^{((\widehat{p}_1,\widehat{p}_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu'_m)}_{(\widehat{p}_2,\dots,\widehat{p}_9)}) + L_m(\zeta_R(\cdot\,; p_1^m, \widehat{p}_1)) + L_m(\zeta_R(\cdot\,; \widehat{p}_{10}, p_{10}^m))\right) \\
&= \sqrt{2}\Bigg(\sum_{j=0}^{4} L_m(y_{P_{k_{j+1}}}(\cdot\,; \widehat{p}_{2j+1}, \widehat{p}_{2j+2}; \varepsilon, \nu'_m)) + \sum_{j=1}^{4} L_m(y_{\mathrm{ext}}(\cdot\,; \widehat{p}_{2j}, \widehat{p}_{2j+1}; \varepsilon, \nu'_m)) \\
&\qquad + L_m(\zeta_R(\cdot\,; p_1^m, \widehat{p}_1)) + L_m(\zeta_R(\cdot\,; \widehat{p}_{10}, p_{10}^m))\Bigg)
\end{aligned} \tag{53}$$

The variational characterization of $y_{P_{k_{j+1}}}(\cdot\,; \widehat{p}_{2j}, \widehat{p}_{2j+1}; \varepsilon, \nu'_m)$ implies that

$$L_m(y_{P_{k_{j+1}}}(\cdot\,; \widehat{p}_{2j+1}, \widehat{p}_{2j+2}; \varepsilon, \nu'_m)) \leq L_m(y_{P_{k_{j+1}}}(\cdot\,; \widehat{p}_{2j+1}, \widehat{p}_{2j+2}; \varepsilon, 0)).$$

Also, let us collect the uniform estimates of equation (43), Lemmas 7.2, 7.3 and 7.4: for every $\lambda > 0$ exists $m_5 := \max\{m_1, \dots, \max\{m_4(\widehat{p}_{2j}, \widehat{p}_{2j+1}) : j = 1, \dots, 4\}\}$ such that

$$\begin{cases} M_m(v_m) > M_0(v_m) - \lambda \\ \sqrt{2}L_m(\zeta_R(\cdot\,; p_1^m, \widehat{p}_1)) \le M_m(\zeta_R(\cdot\,; p_1^m, \widehat{p}_1)) < 2\lambda \\ \sqrt{2}L_m(\zeta_R(\cdot\,; \widehat{p}_{10}, p_{10}^m)) \le M_m(\zeta_R(\cdot\,; \widehat{p}_{10}, p_{10}^m)) < 2\lambda \\ L_m(y_{\mathrm{ext}}(\cdot\,; \widehat{p}_{2j}, \widehat{p}_{2j+1}; \varepsilon, \nu'_m)) < L_m(y_{\mathrm{ext}}(\cdot\,; \widehat{p}_{2j}, \widehat{p}_{2j+1}; \varepsilon, 0)) + \lambda \\ M_m(v_0) < M_0(v_0) + \lambda \end{cases}$$

for every $m > m_5$. Therefore, for every $\lambda > 0$ the chain of inequalities (53) gives

$$M_0(\sigma_m) - \lambda \le \sqrt{2}\left(\sum_{j=0}^{4} L_m(y_{P_{k_{j+1}}}(\cdot\,; \widehat{p}_{2j+1}, \widehat{p}_{2j+2}; \varepsilon, 0)) + \sum_{j=1}^{4} L_m(y_{\mathrm{ext}}(\cdot\,; \widehat{p}_{2j}, \widehat{p}_{2j+1}; \varepsilon, 0))\right)$$
$$+ (1+\sqrt{2})4\lambda = \sqrt{2}L_m(\sigma_0) + C\lambda \le M_m(\sigma_0) + C\lambda$$

if $m > m_5$. With a change of variable, we can see that the previous inequality is equivalent to

$$M_0(v_m) - \lambda \le M_m(v_0) + C\lambda \Rightarrow M_0(v_m) \le M_0(v_0) + (C+1)\lambda$$

if $m > m_5$; since λ has been arbitrarily chosen, it results $\limsup_m M_0(v_m) \le M_0(v_0)$; comparing with (52) we deduce that $M_0(v_0) = M_0(\widetilde{v})$, and the proof is complete. $\square$

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